Theory and application of a computational framework for crystal plasticity

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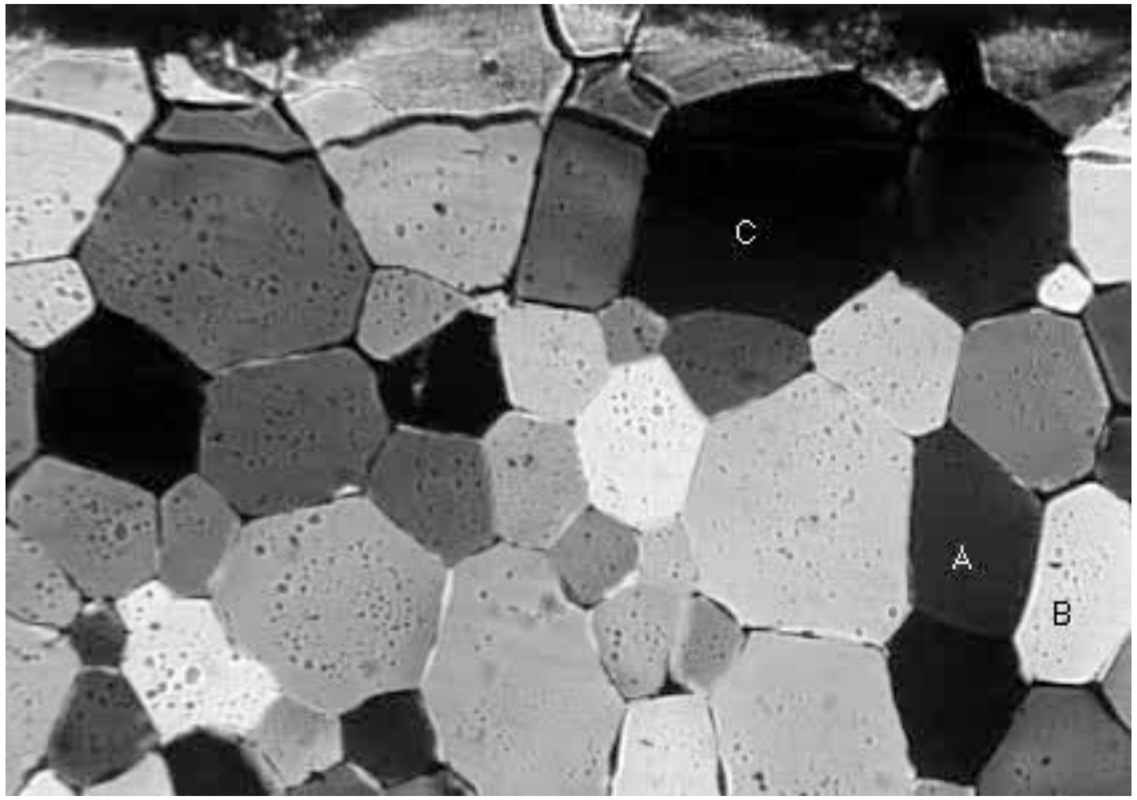
goal

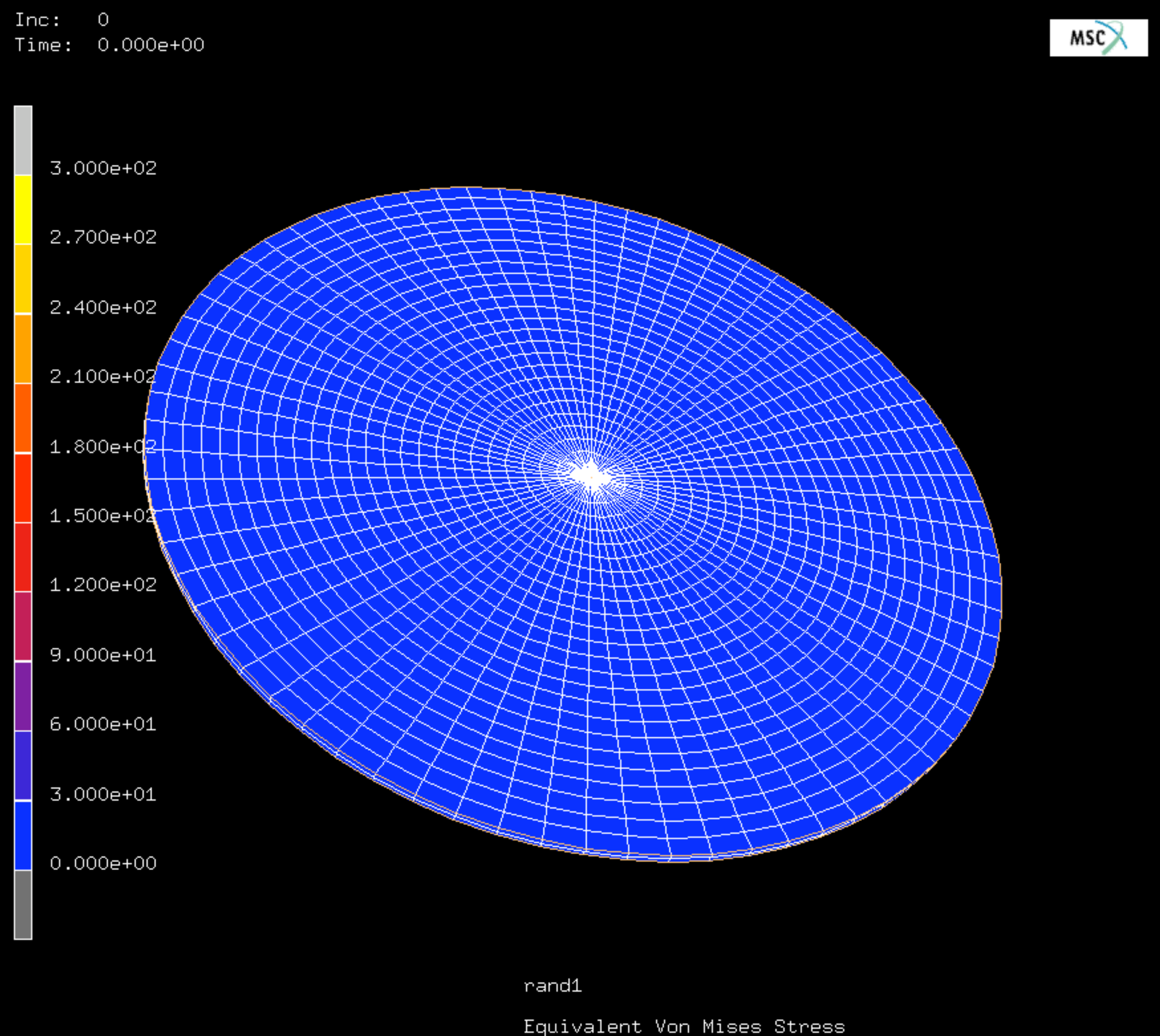
predicting the
solid mechanics of metals
in structural applications

predicting the solid mechanics of metals in structural applications

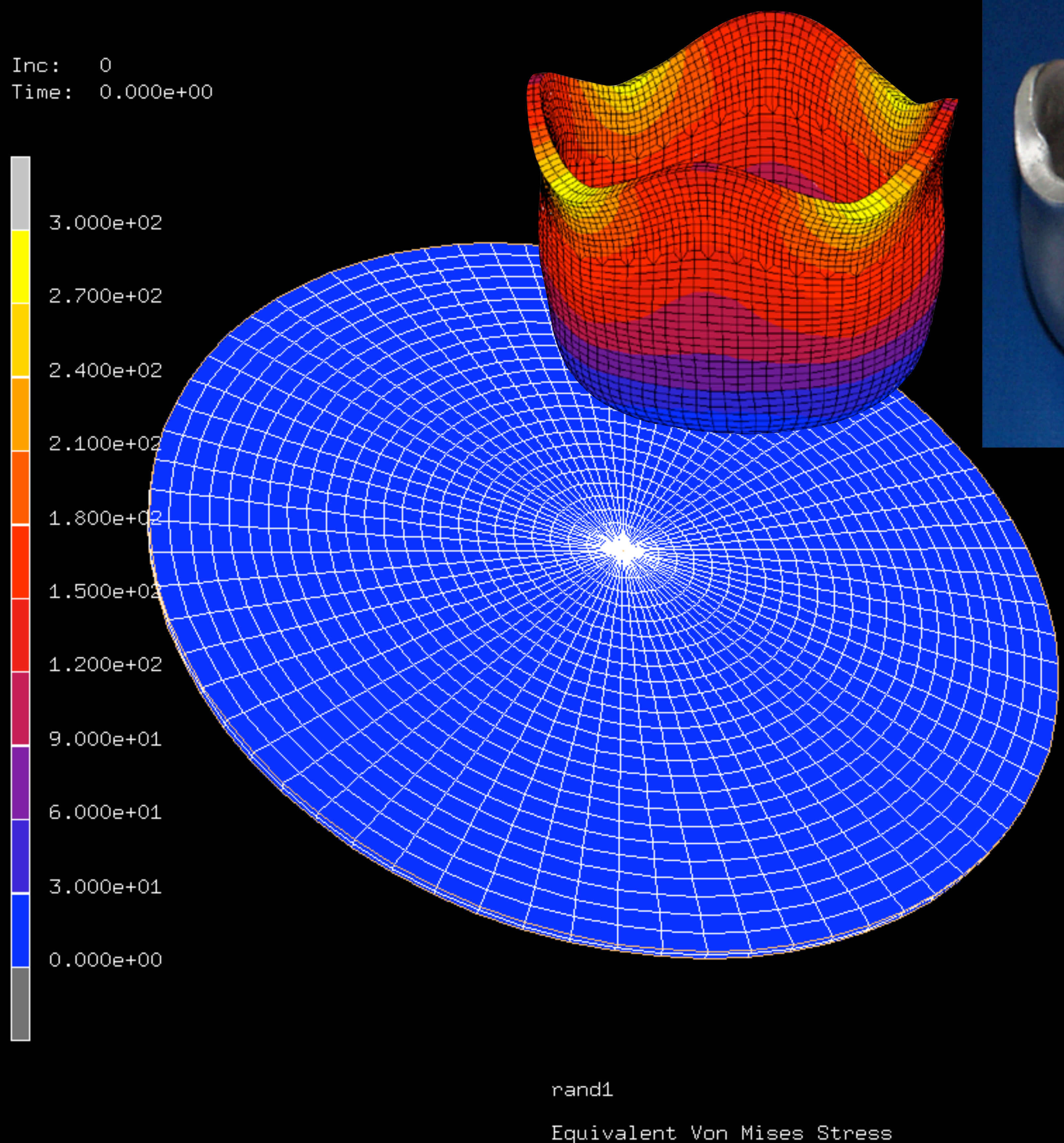


from http://courses.eas.ualberta.ca/eas421/lecturepages/microstructurediags.html

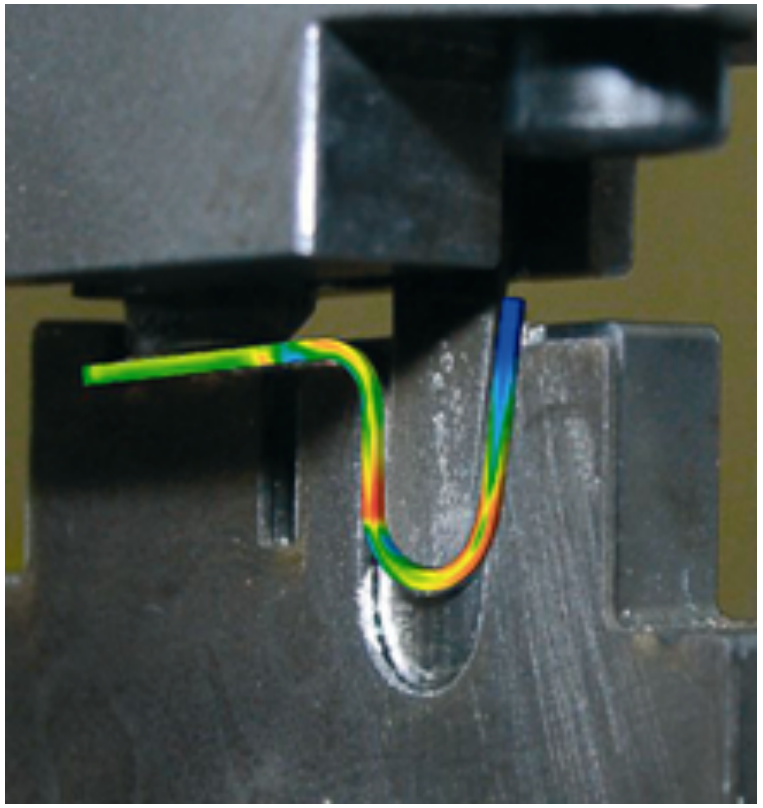
deep drawing simulation



deep drawing simulation



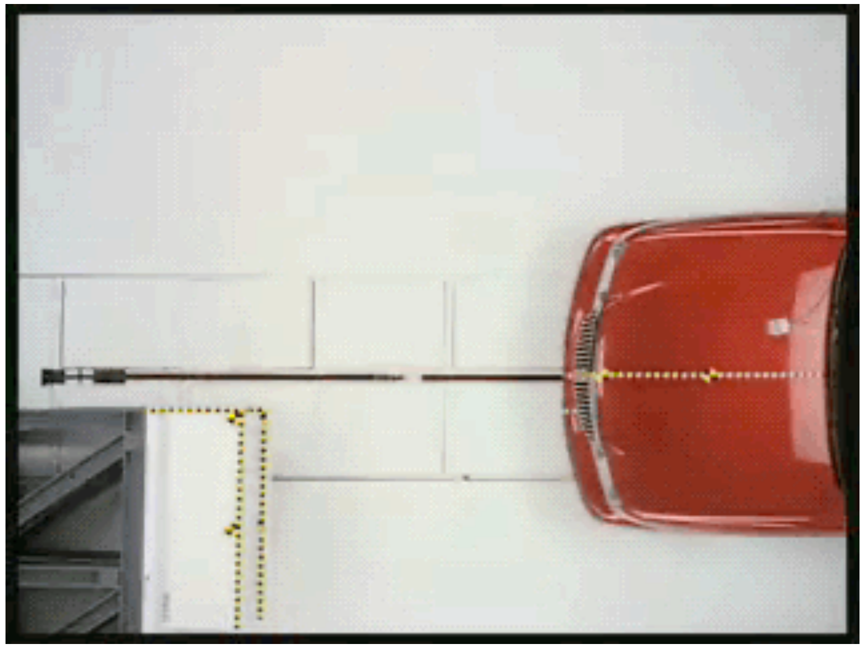
examples

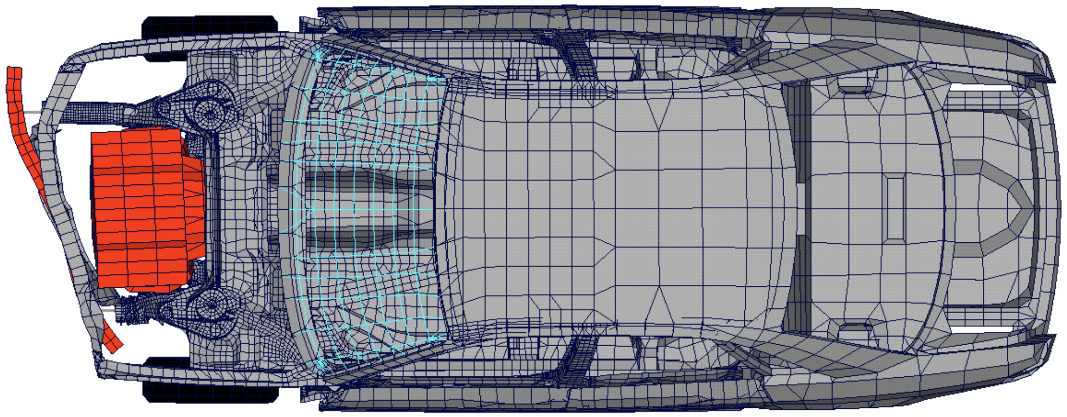


VAW aluminium AG

University Gent, Belgium

goal



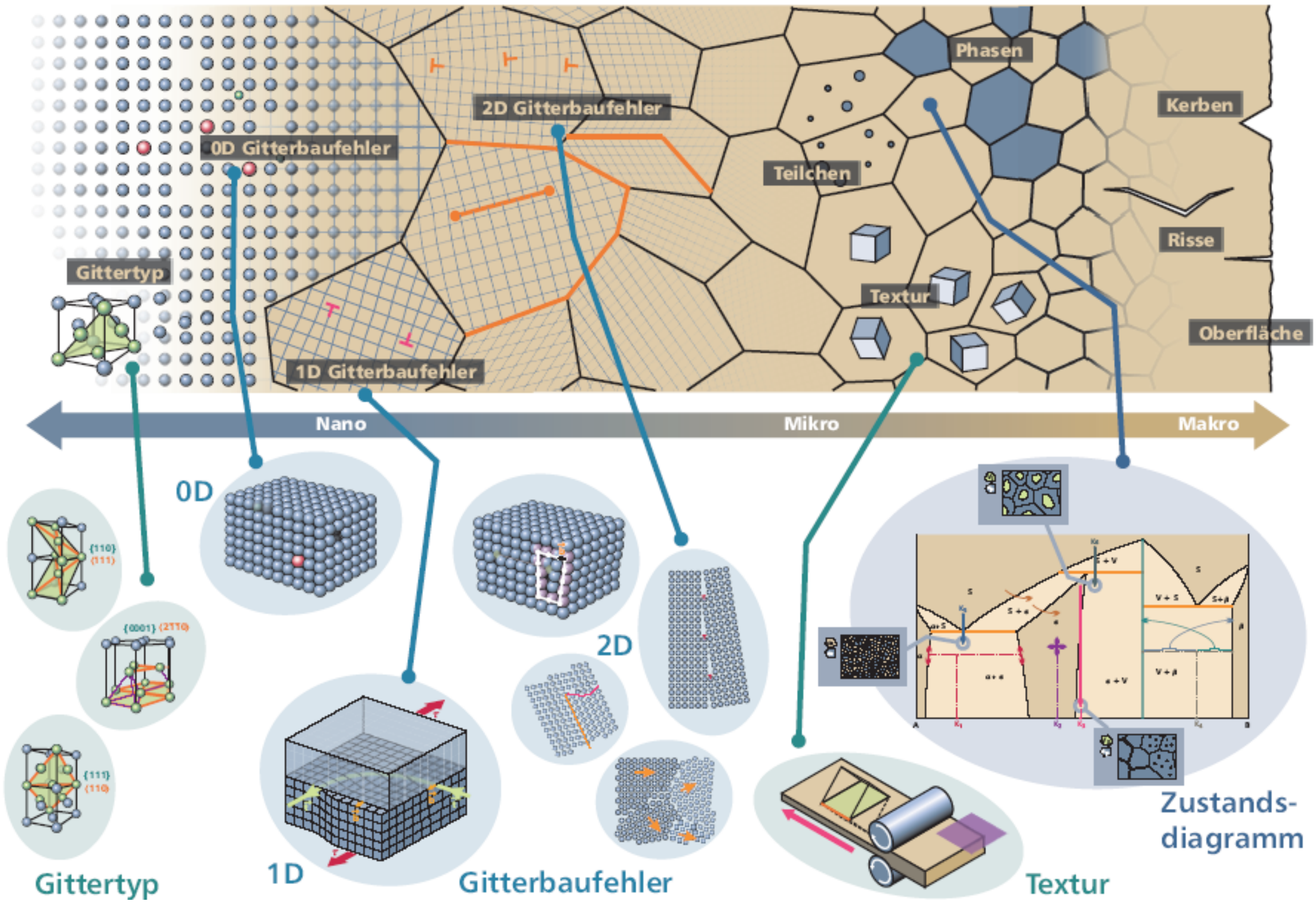


PART I

Background on

Crystalline Solids

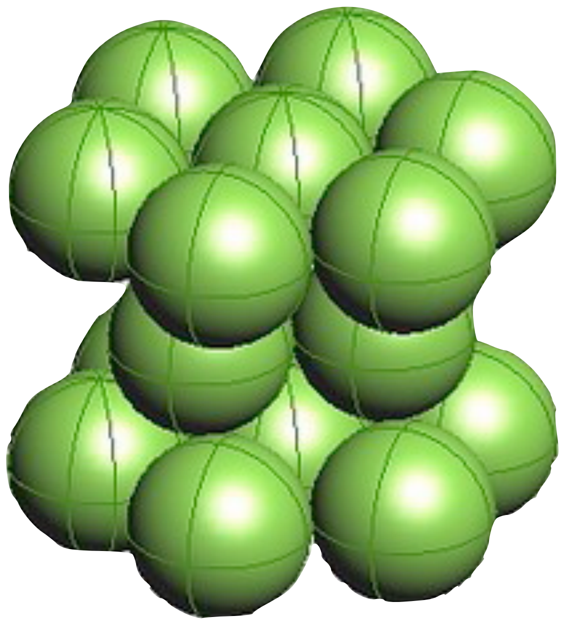
crystalline solid



crystal lattice structures

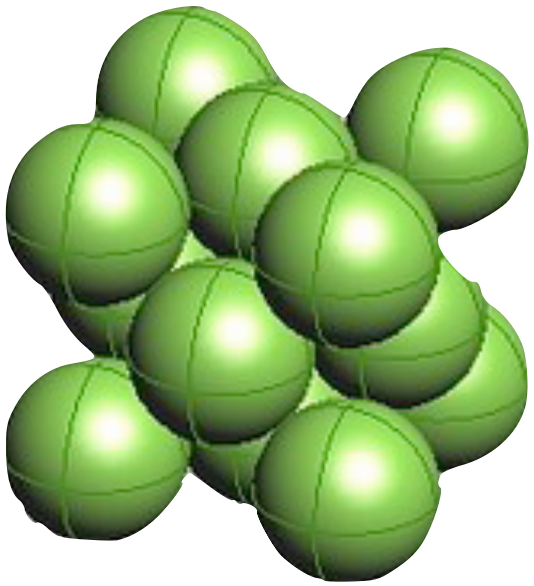
face
centered
cubic

body
centered
cubic

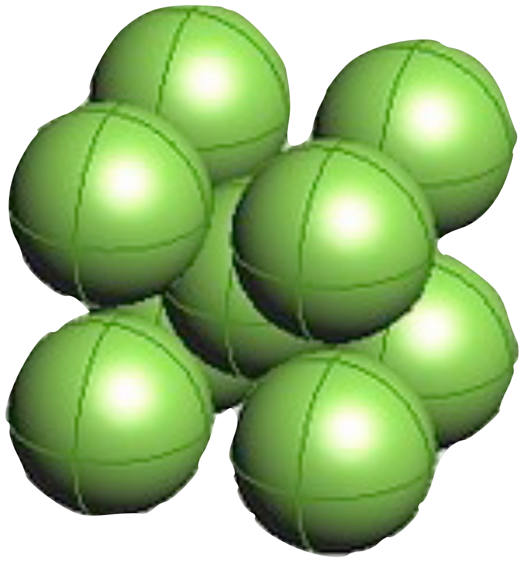


hexagonal
close packed

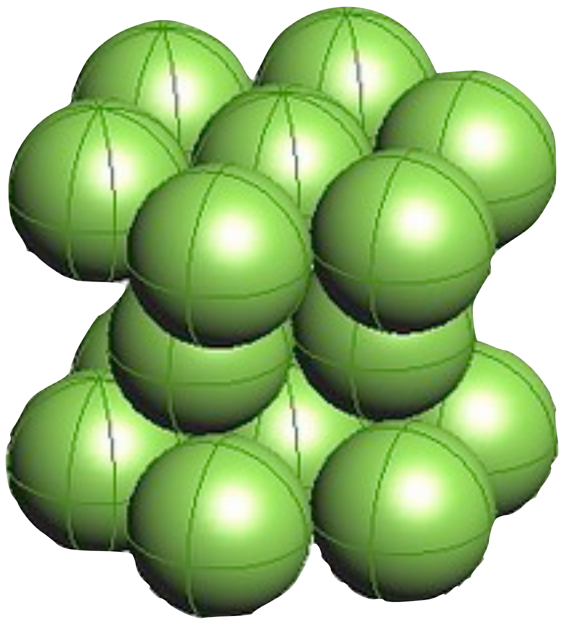
crystal lattice structures



face
centered
cubic

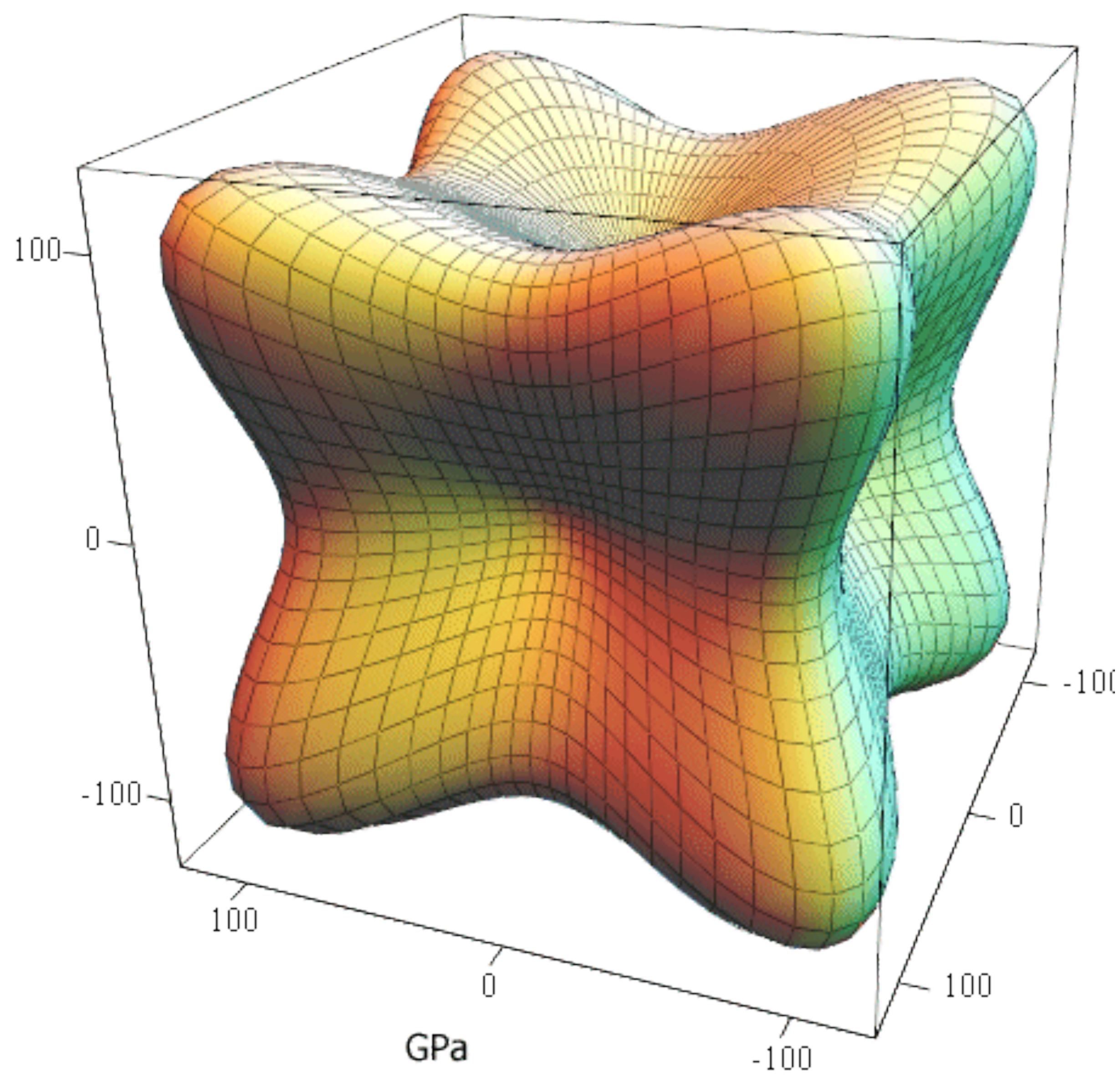


body
centered
cubic

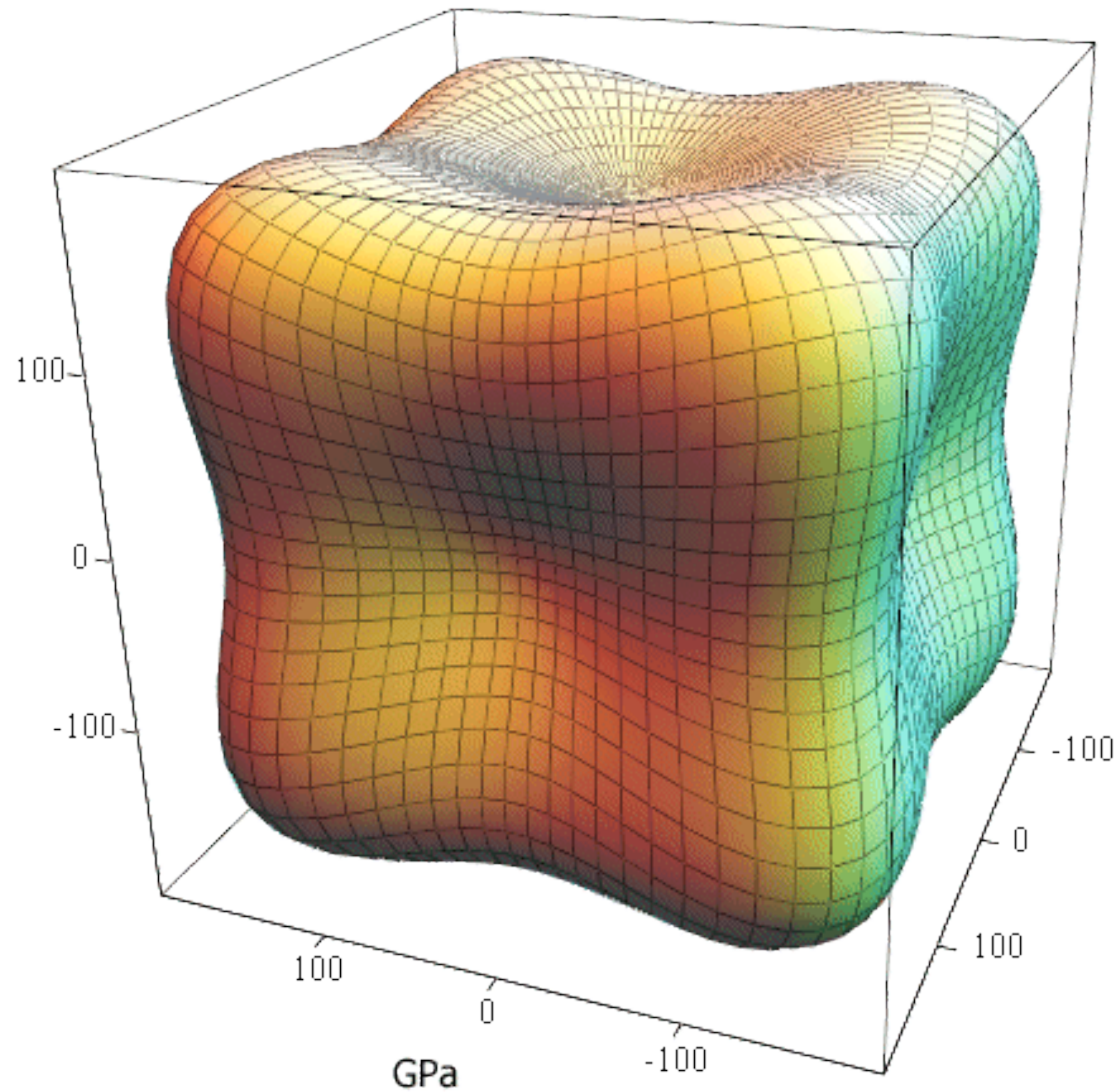


hexagonal
close packed

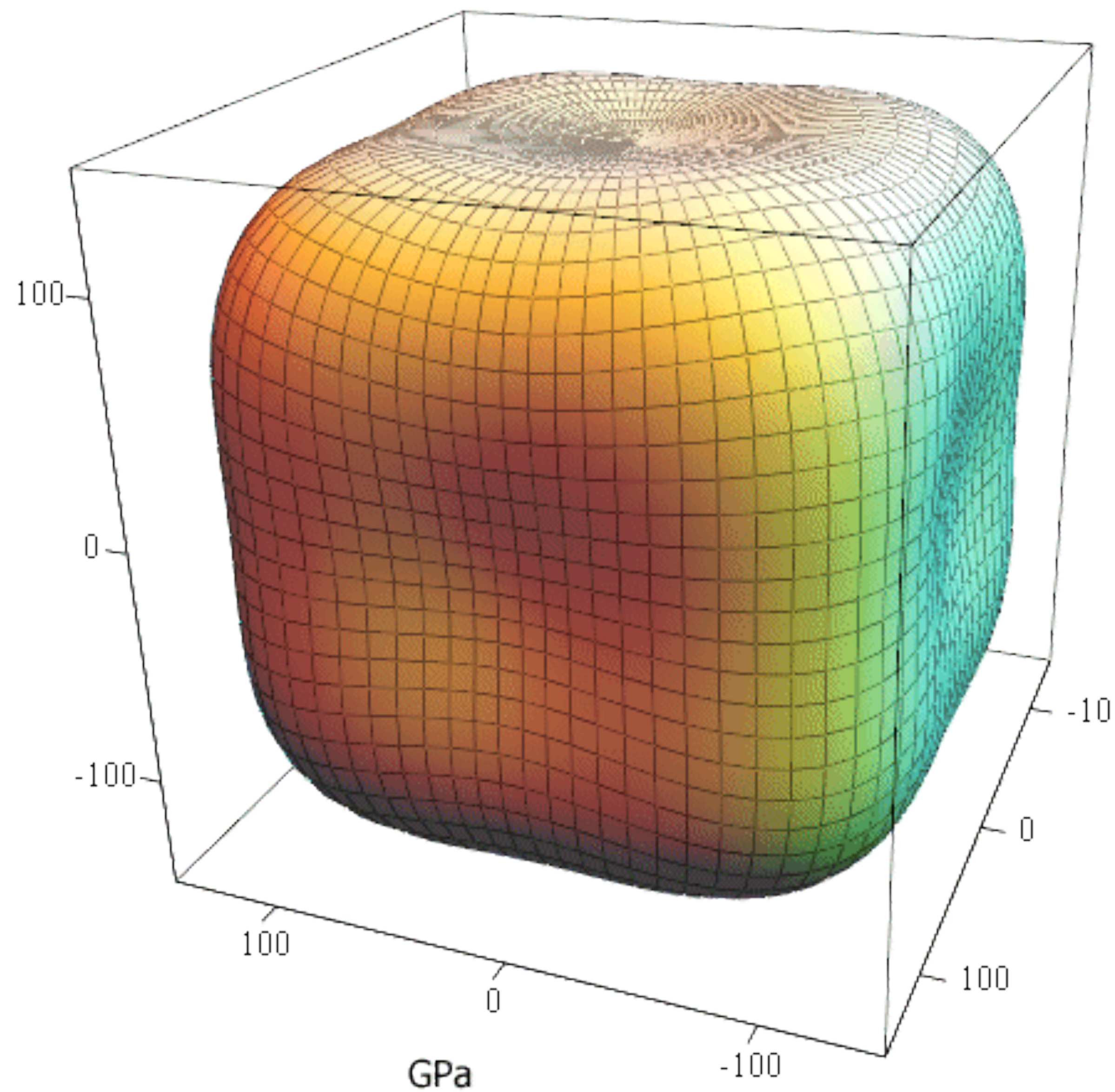
anisotropy of elastic stiffness



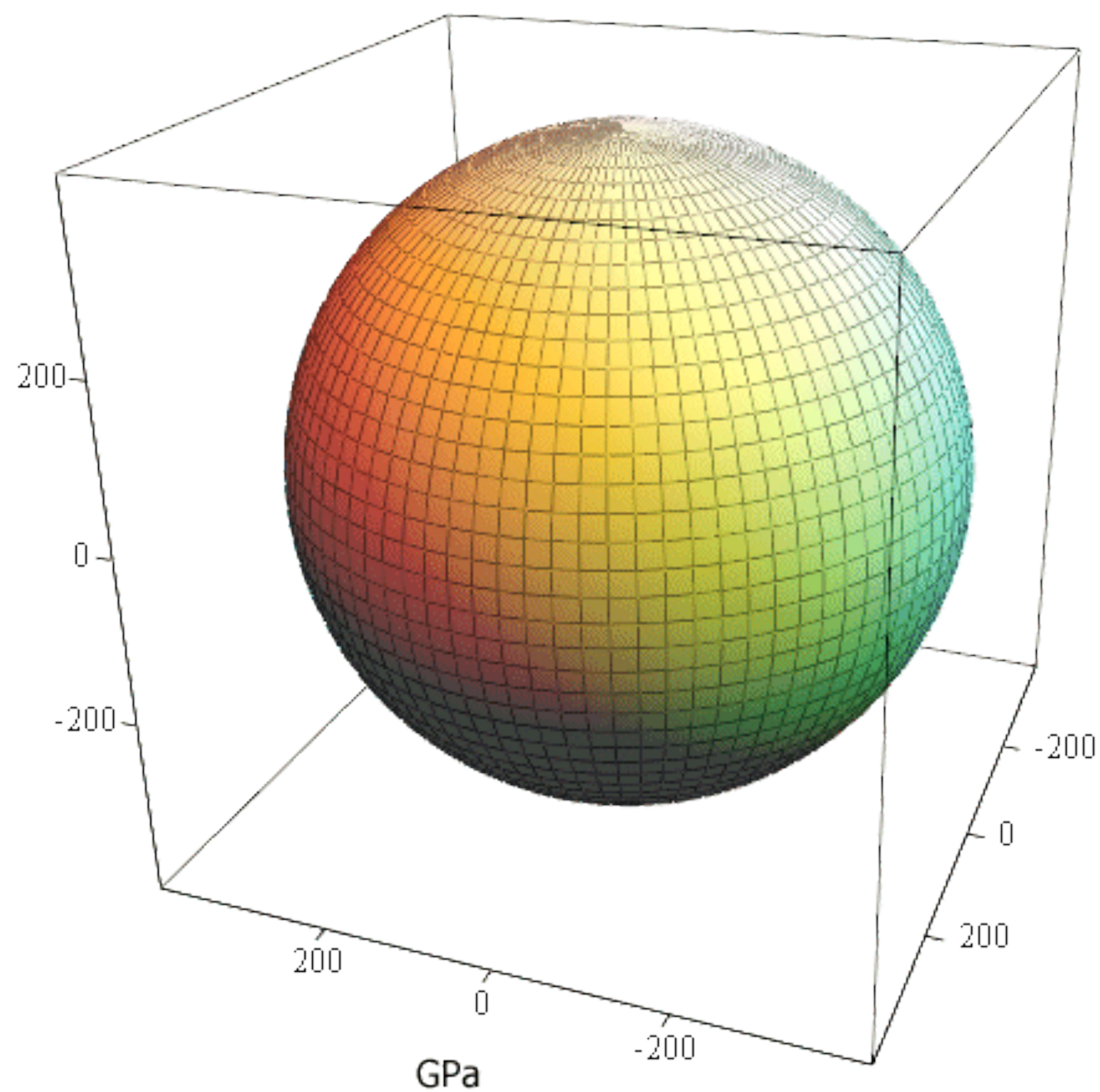
anisotropy of elastic stiffness



anisotropy of elastic stiffness



anisotropy of elastic stiffness

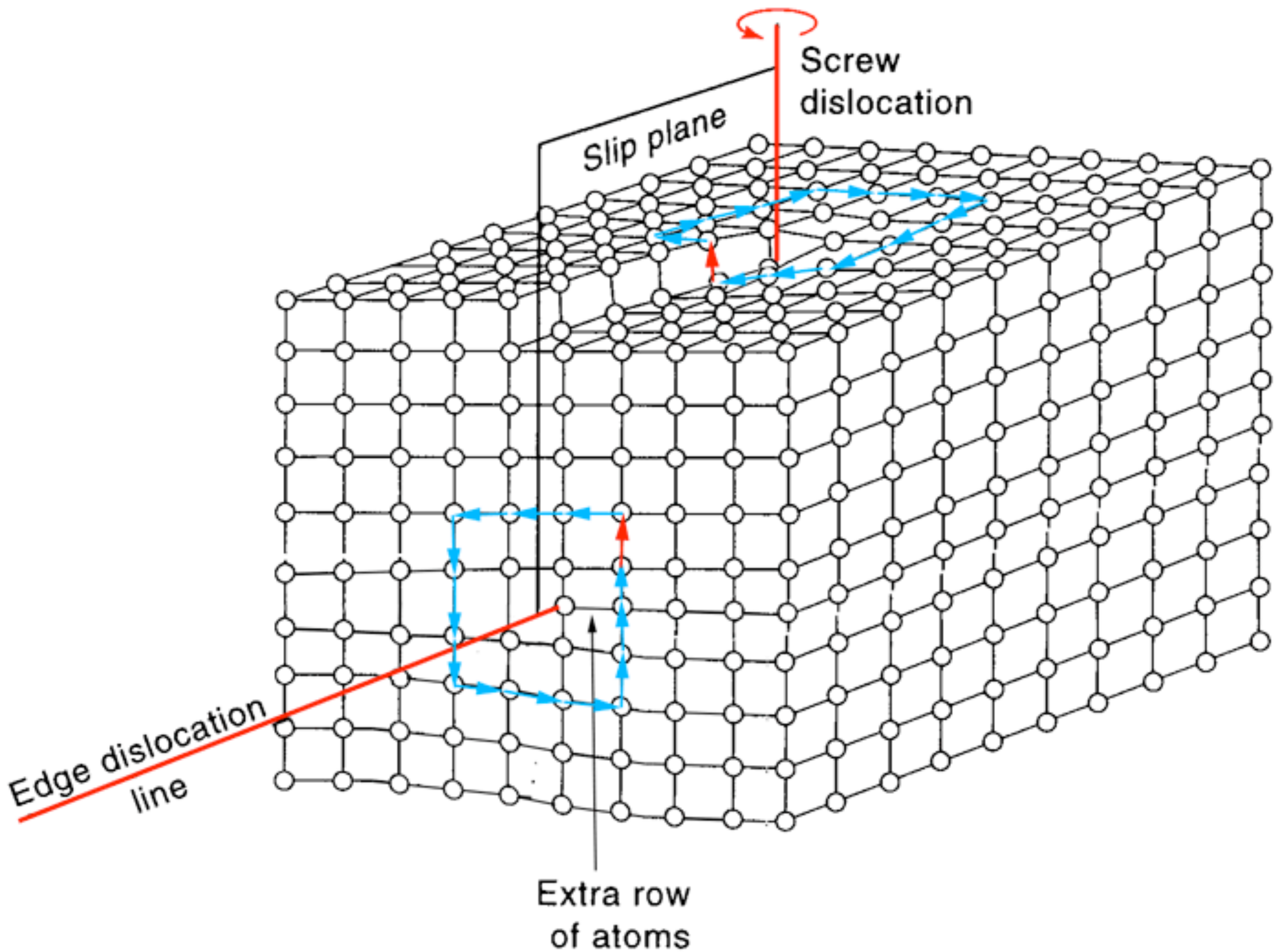


plastic deformation of monocrystals

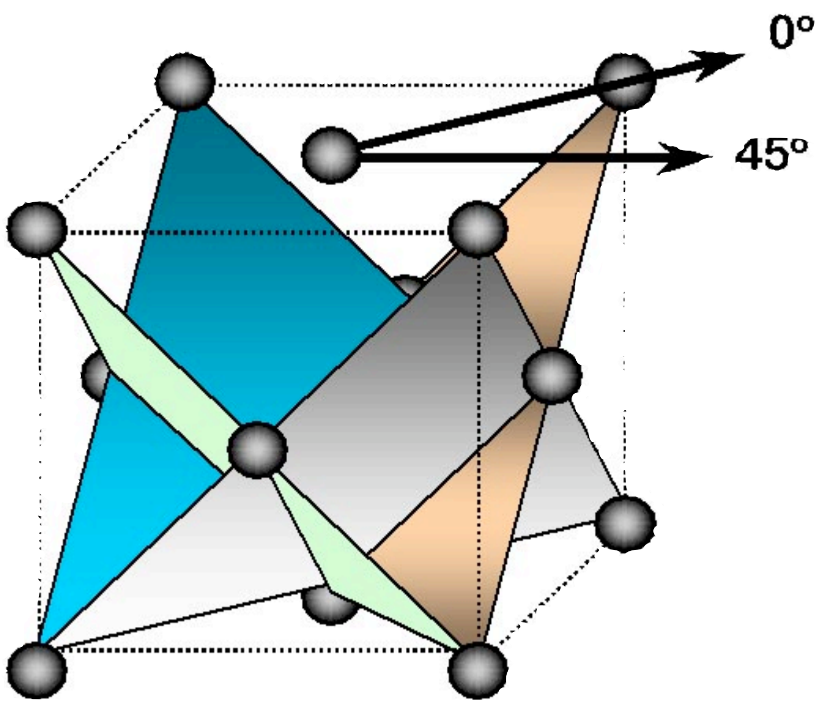
inherently anisotropic due to underlying mechanisms:

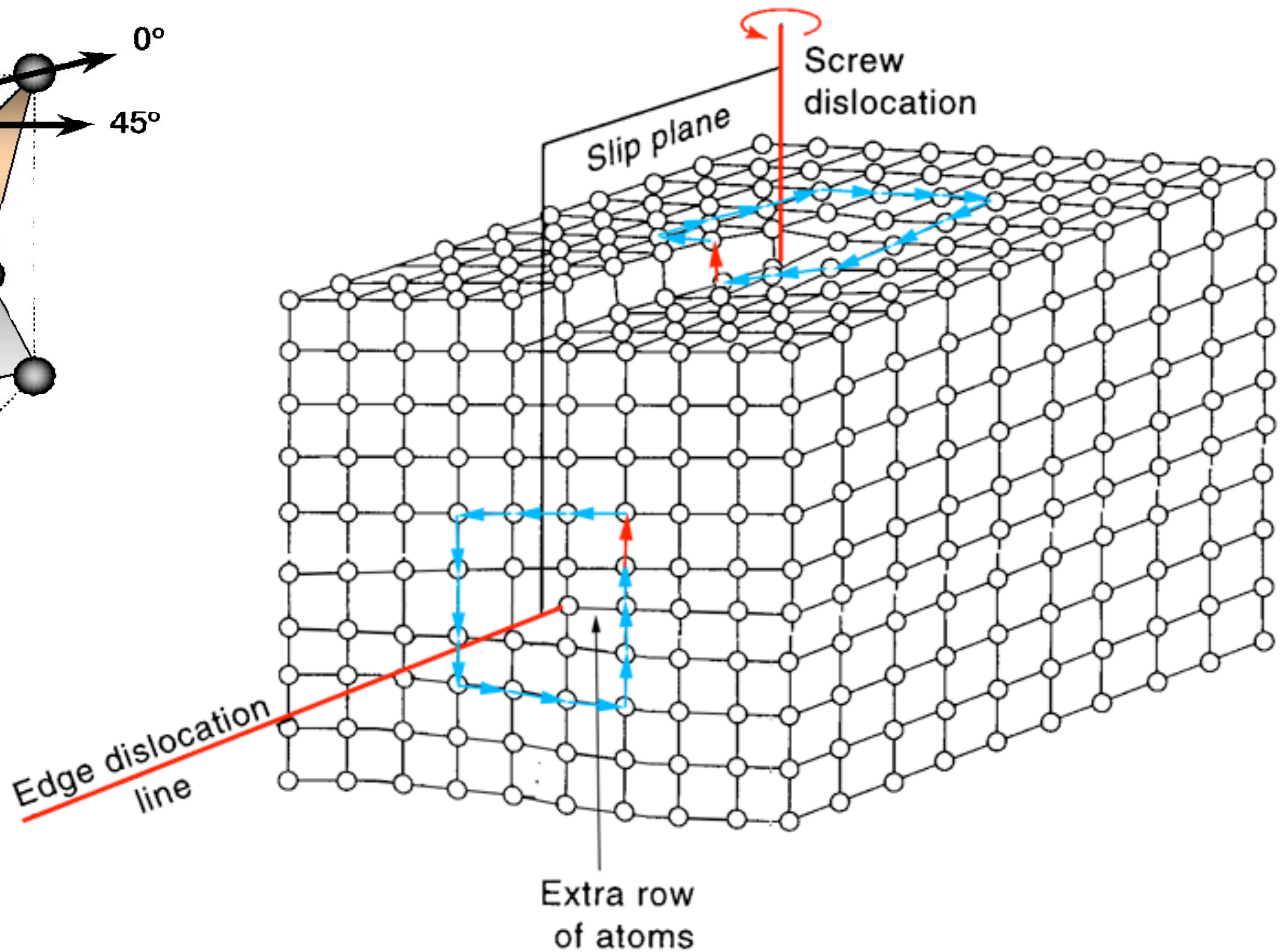
- dislocation slip
- mechanical twinning
- displacive phase transformation

dislocation

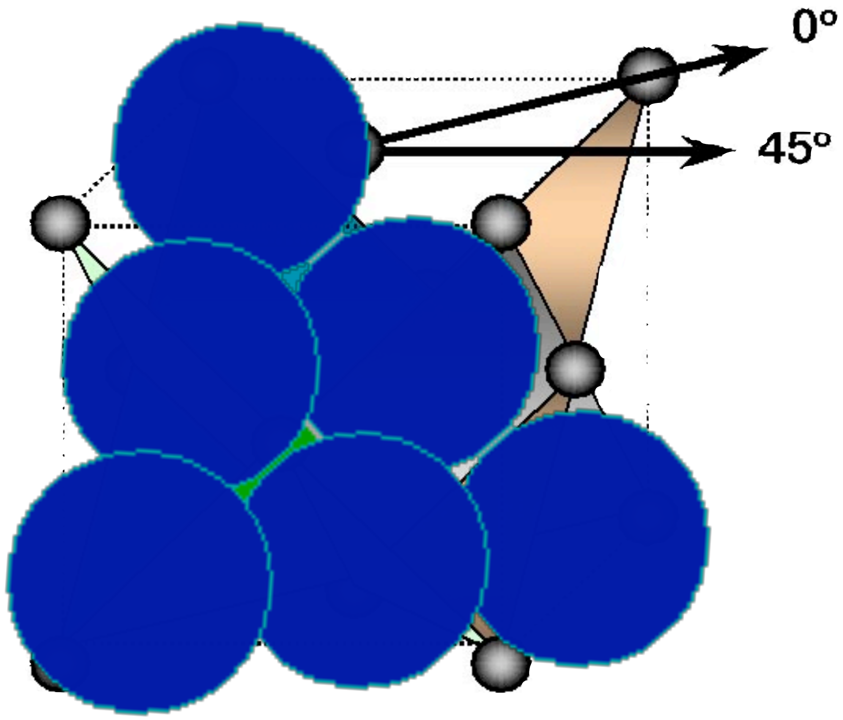


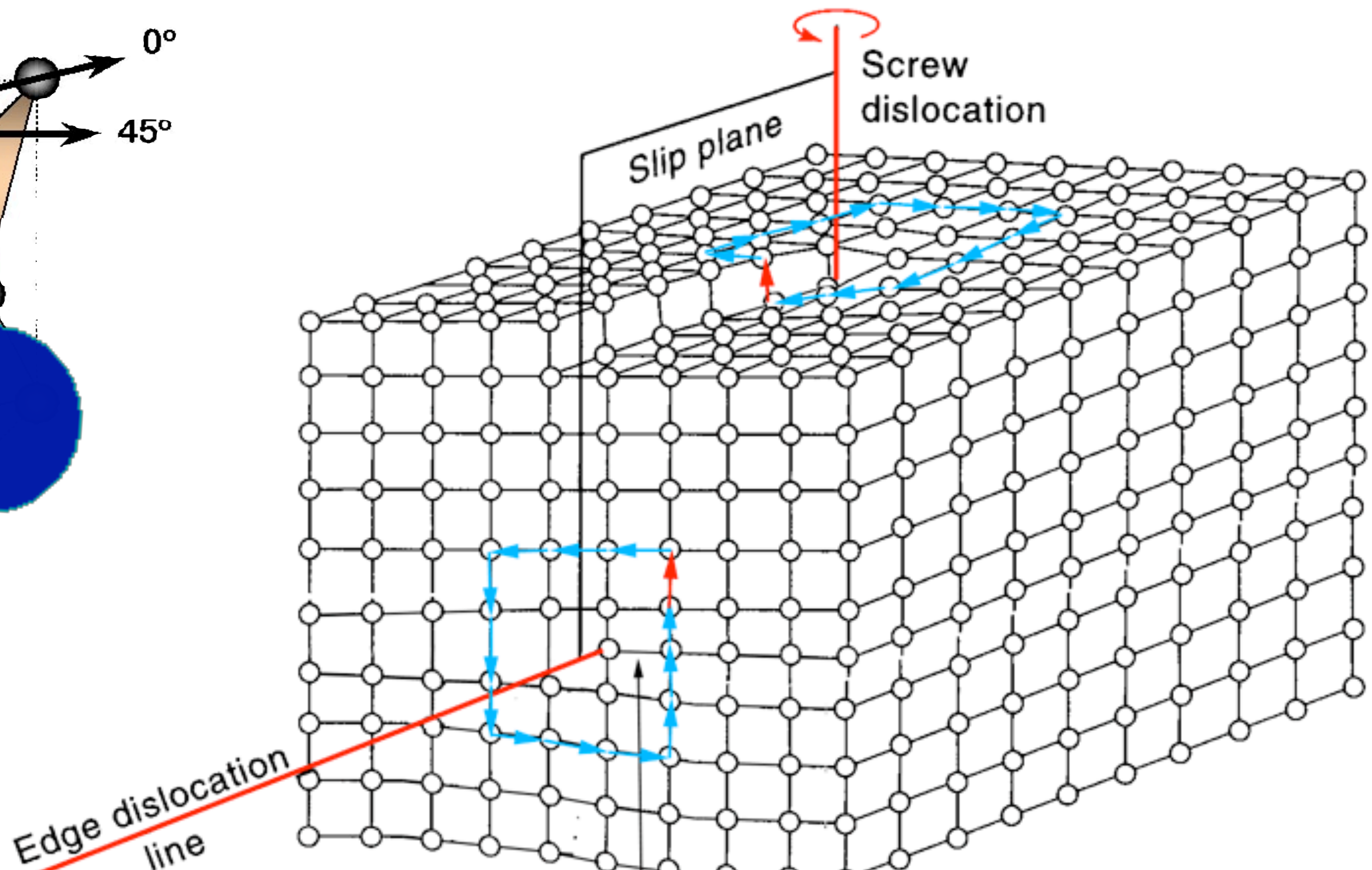
dislocation



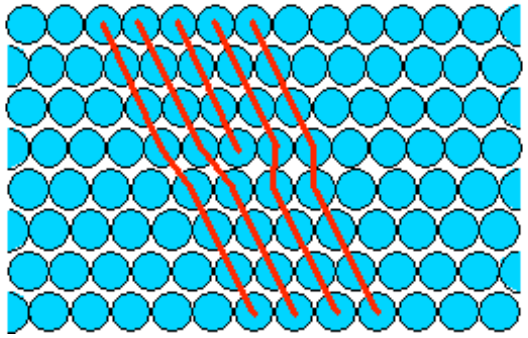


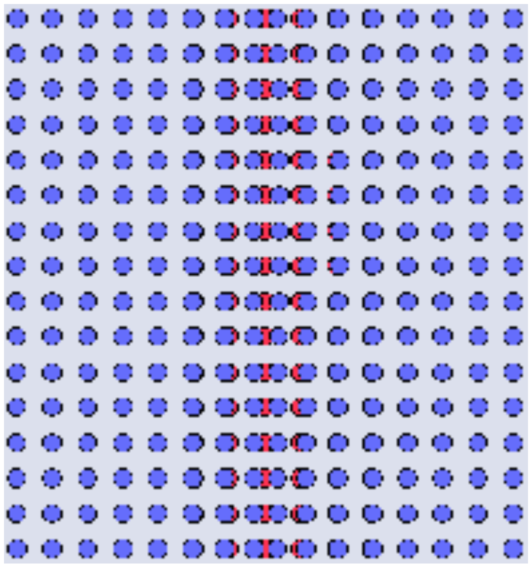
dislocation

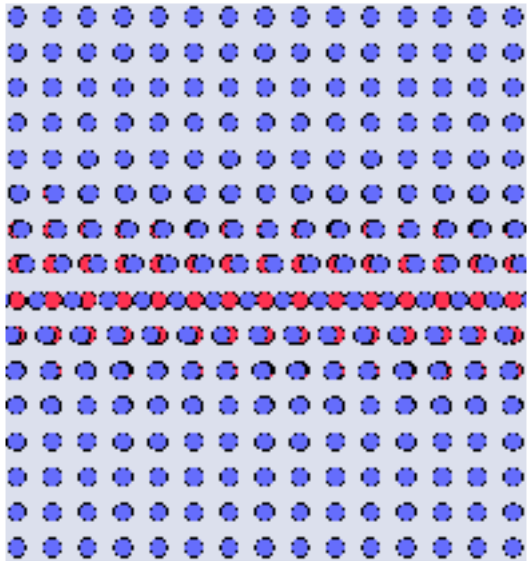


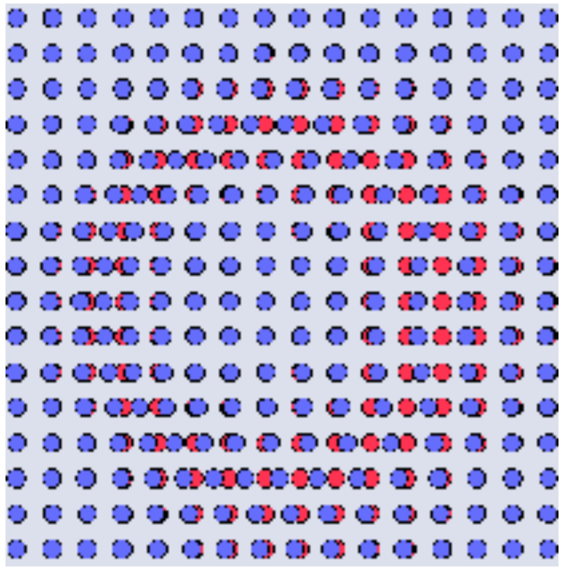


dislocation

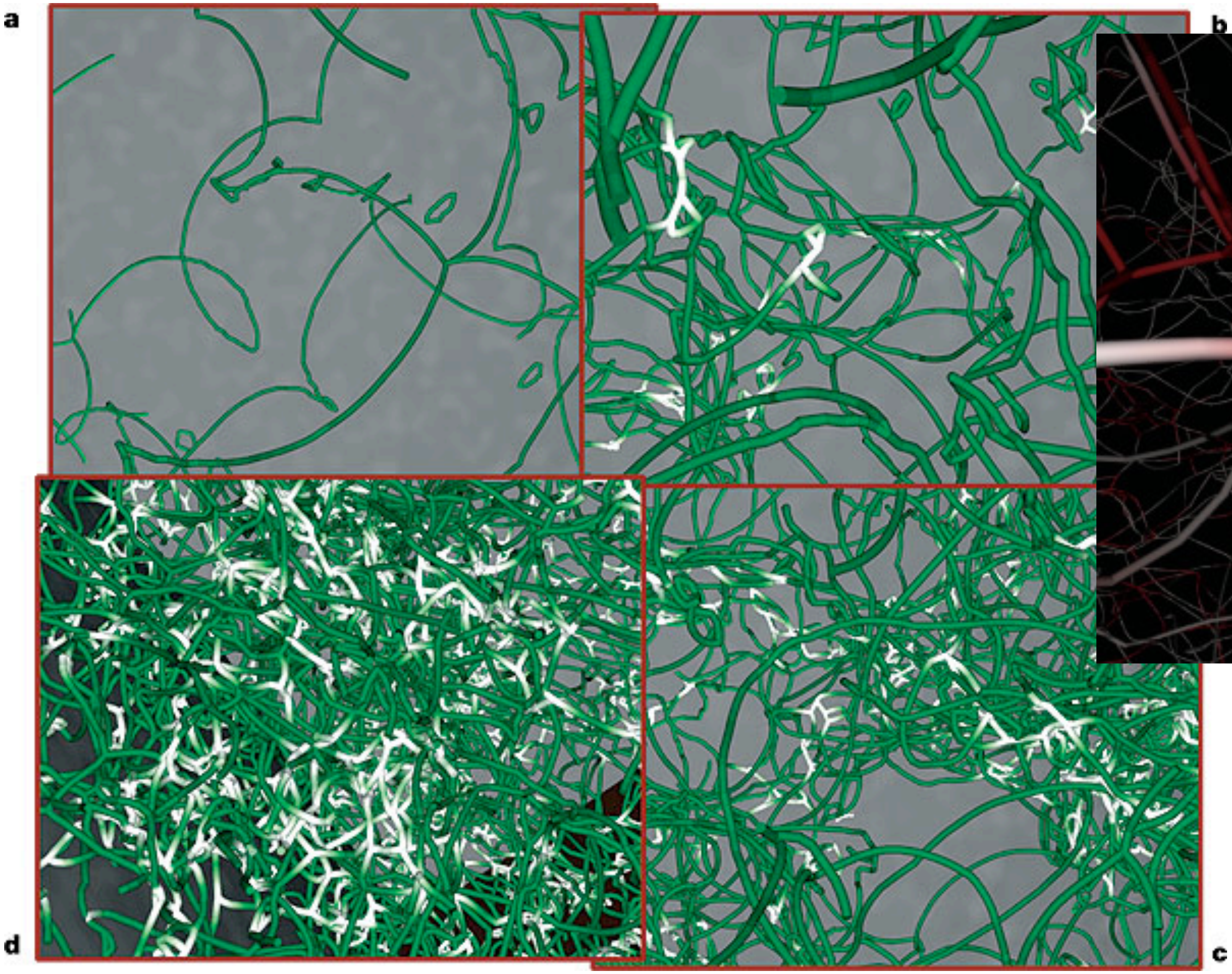


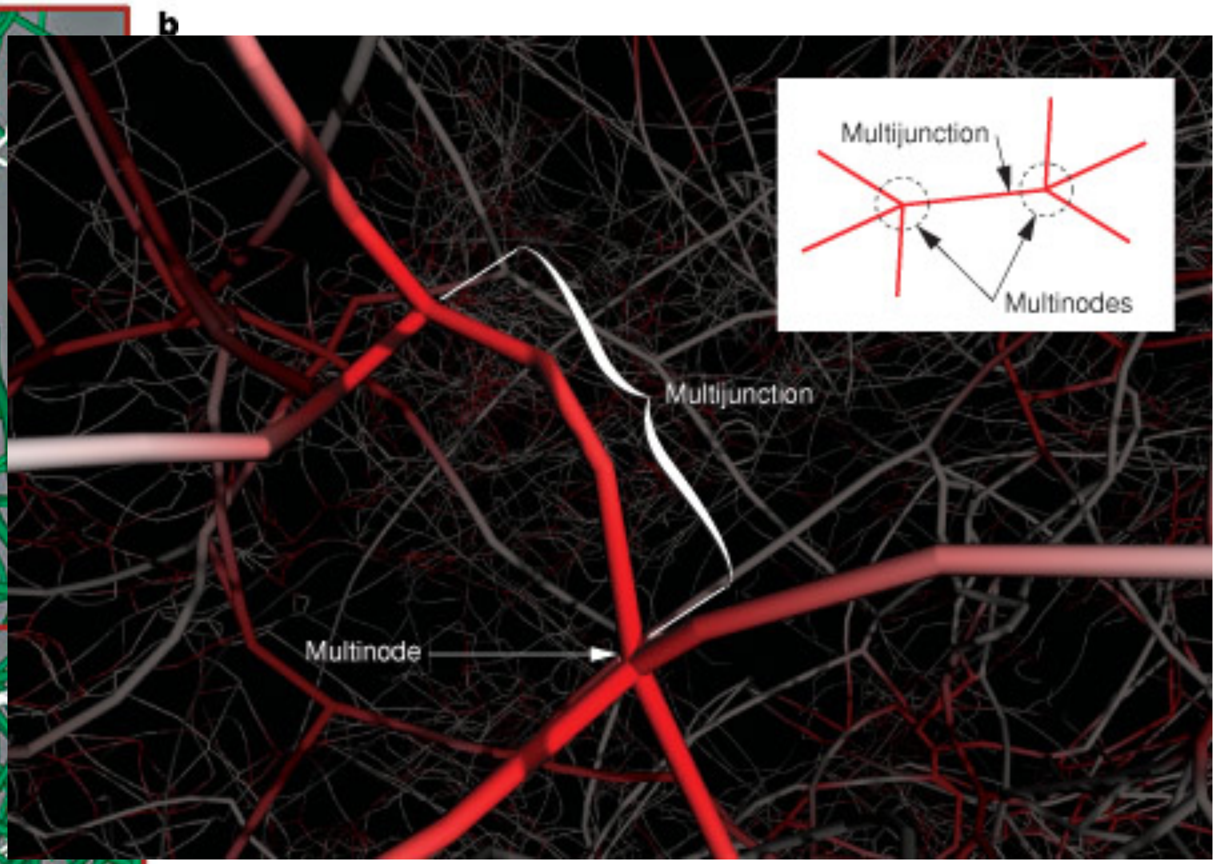






dislocation – dislocation interaction

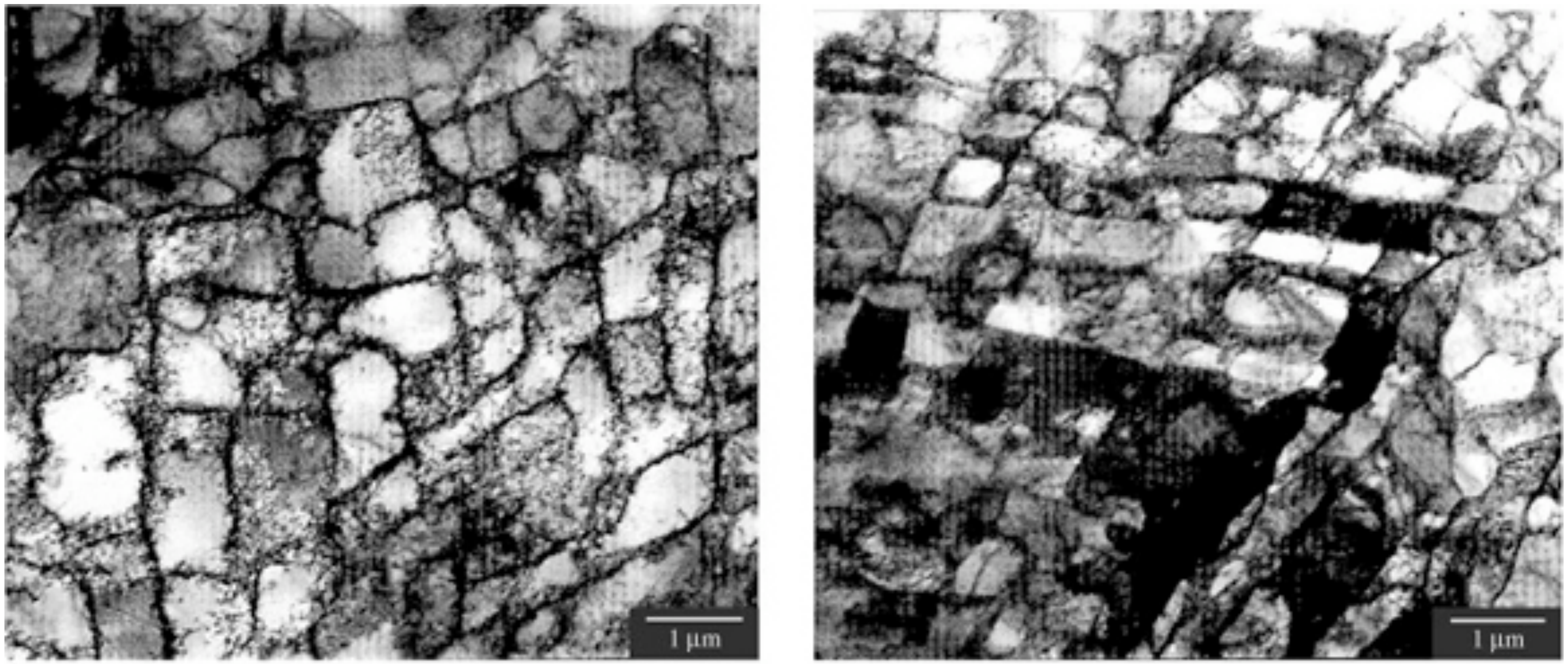




from DOI:10.1038/nature04658

from http://www-cmls.llnl.gov

dislocation structure

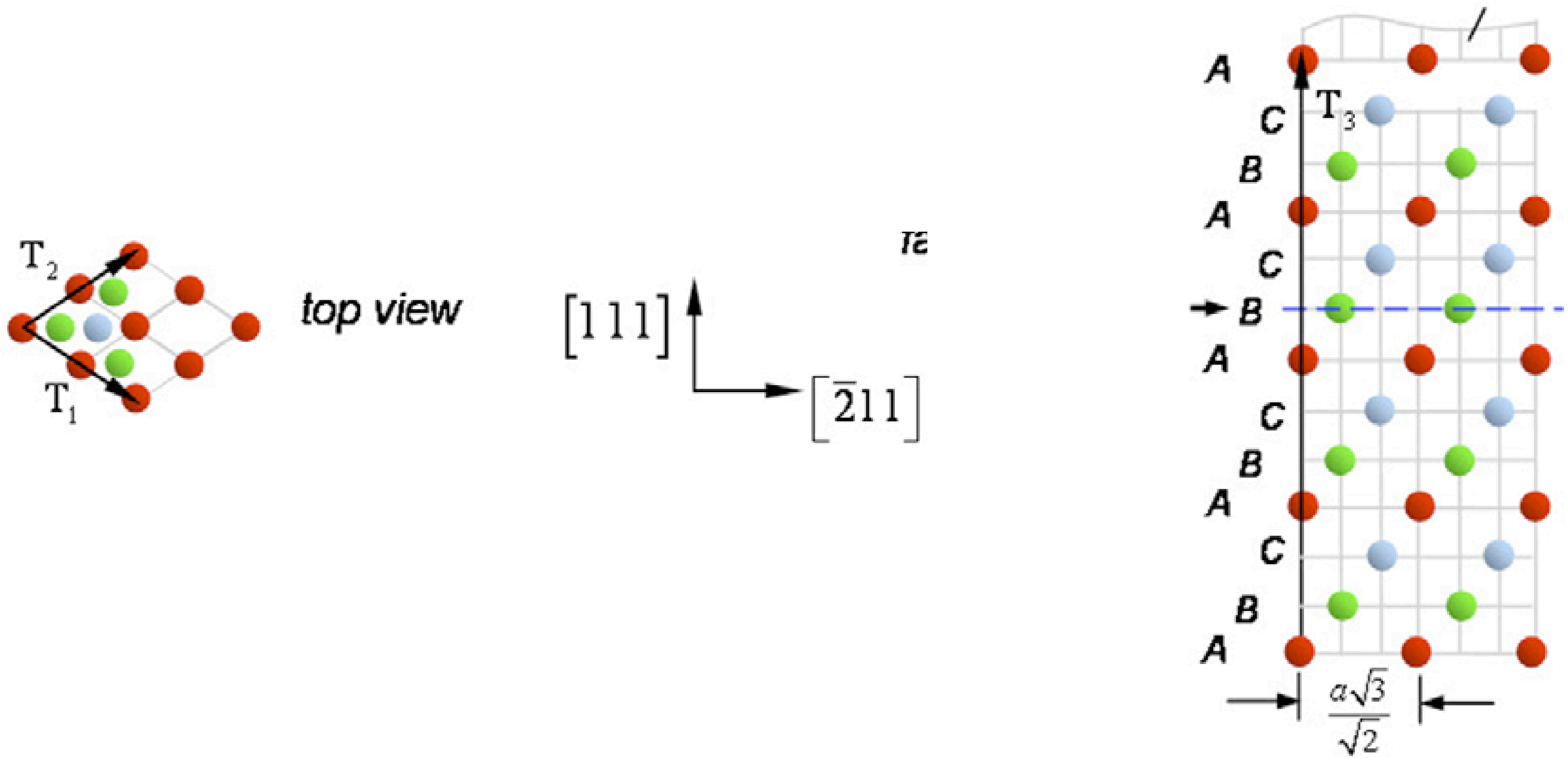


(a) (b)

Figure 5. TEM of different regions of steel drawn in 2 passes (8° e 20%), and cyclically twisted (11.2% per cycle, 10 cycles).

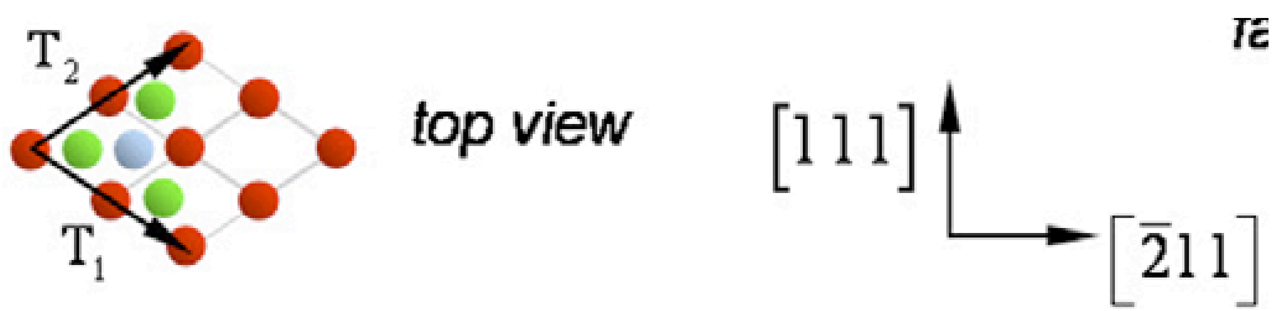
from DOI:10.1590/S1516-14392006000300016

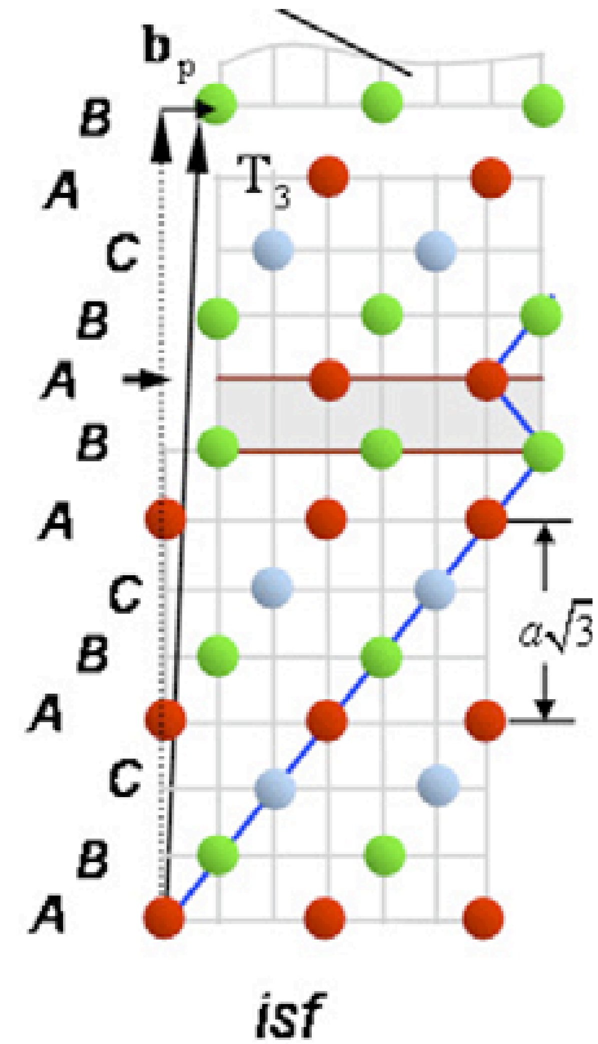
mechanical twinning



from S. Kibey et al., Acta Mater. 55 (2007) p. 6843

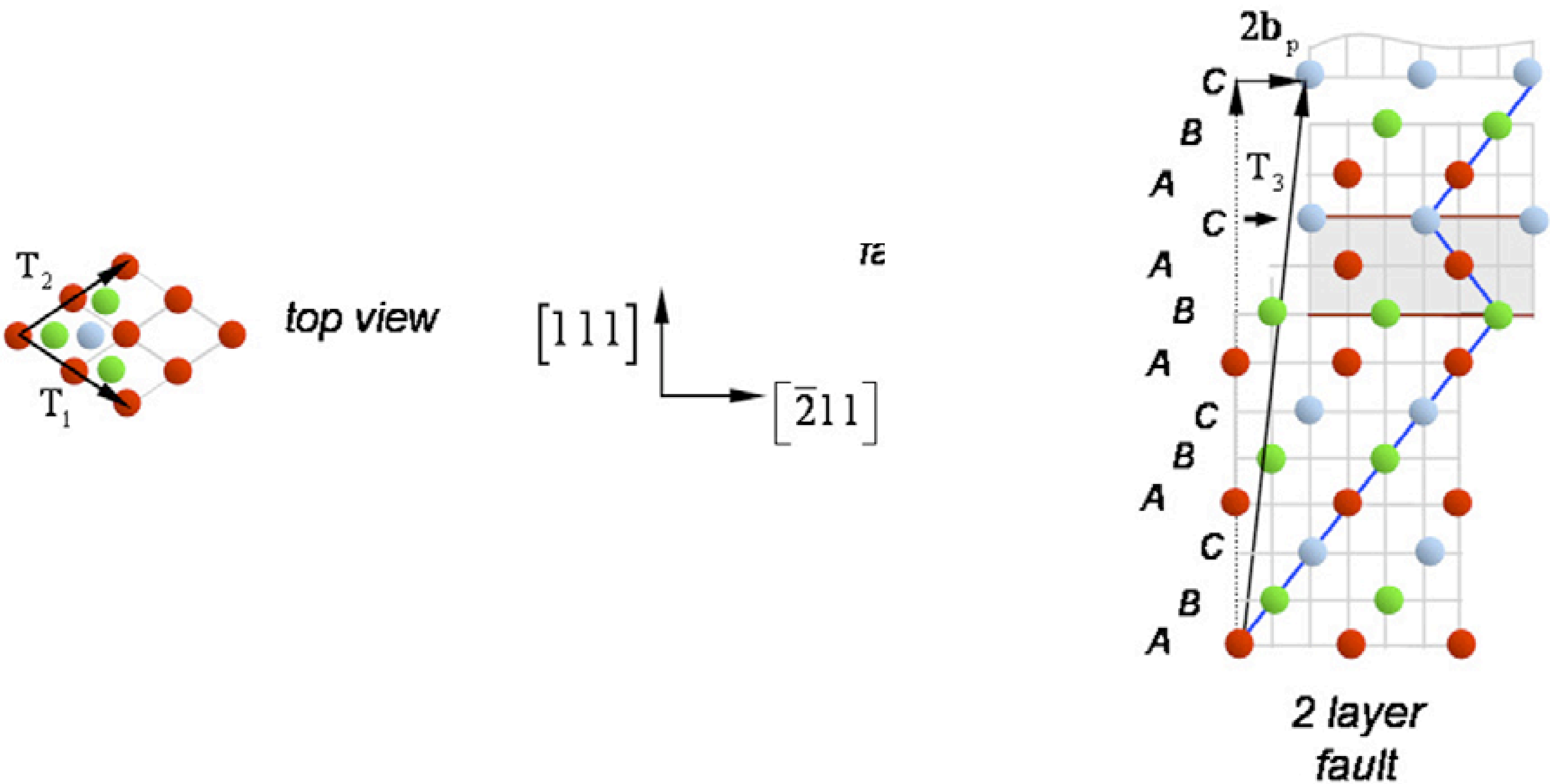
mechanical twinning





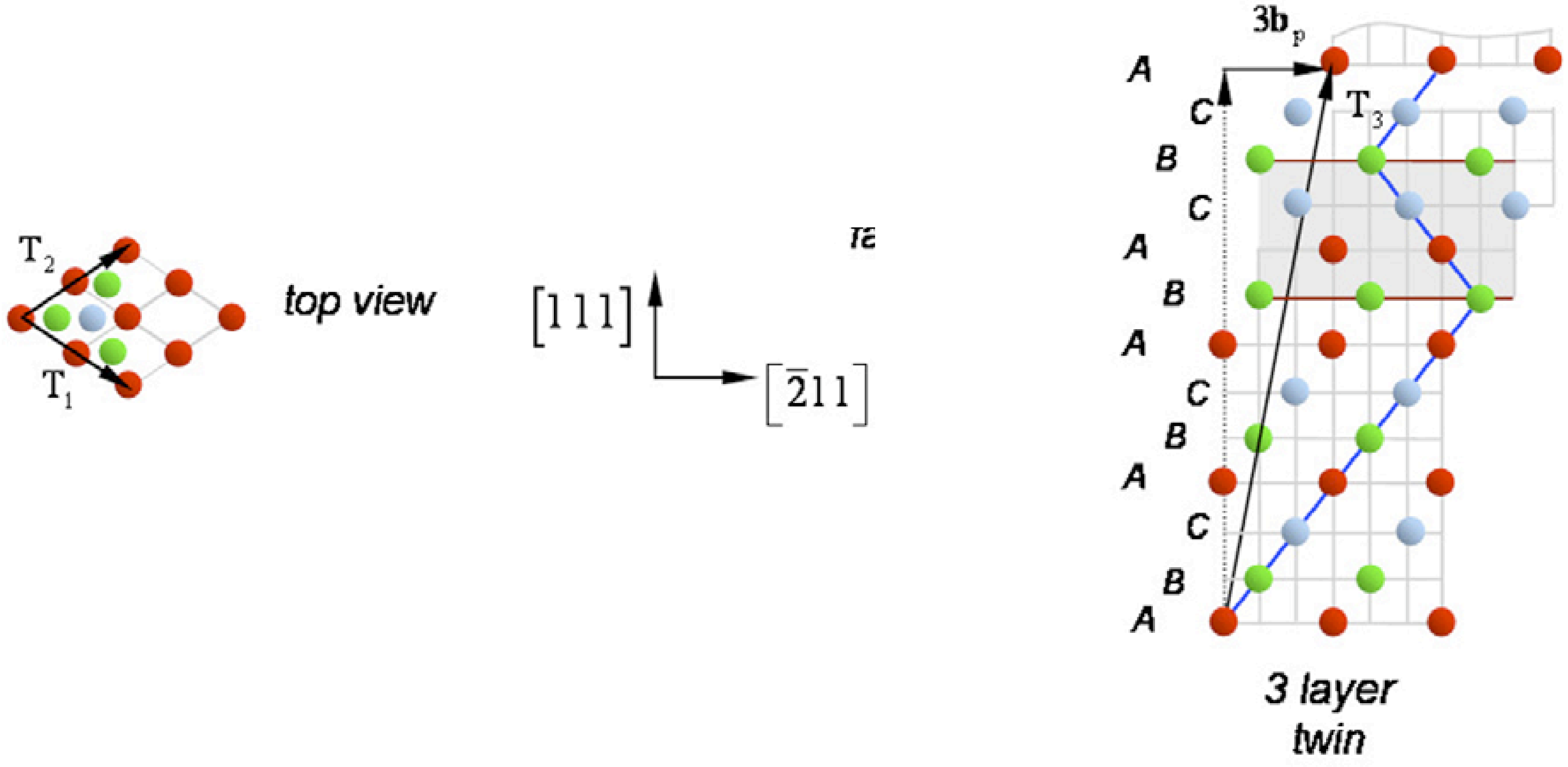
from S. Kibey et al., Acta Mater. 55 (2007) p. 6843

mechanical twinning



from S. Kibey et al., Acta Mater. 55 (2007) p. 6843

mechanical twinning



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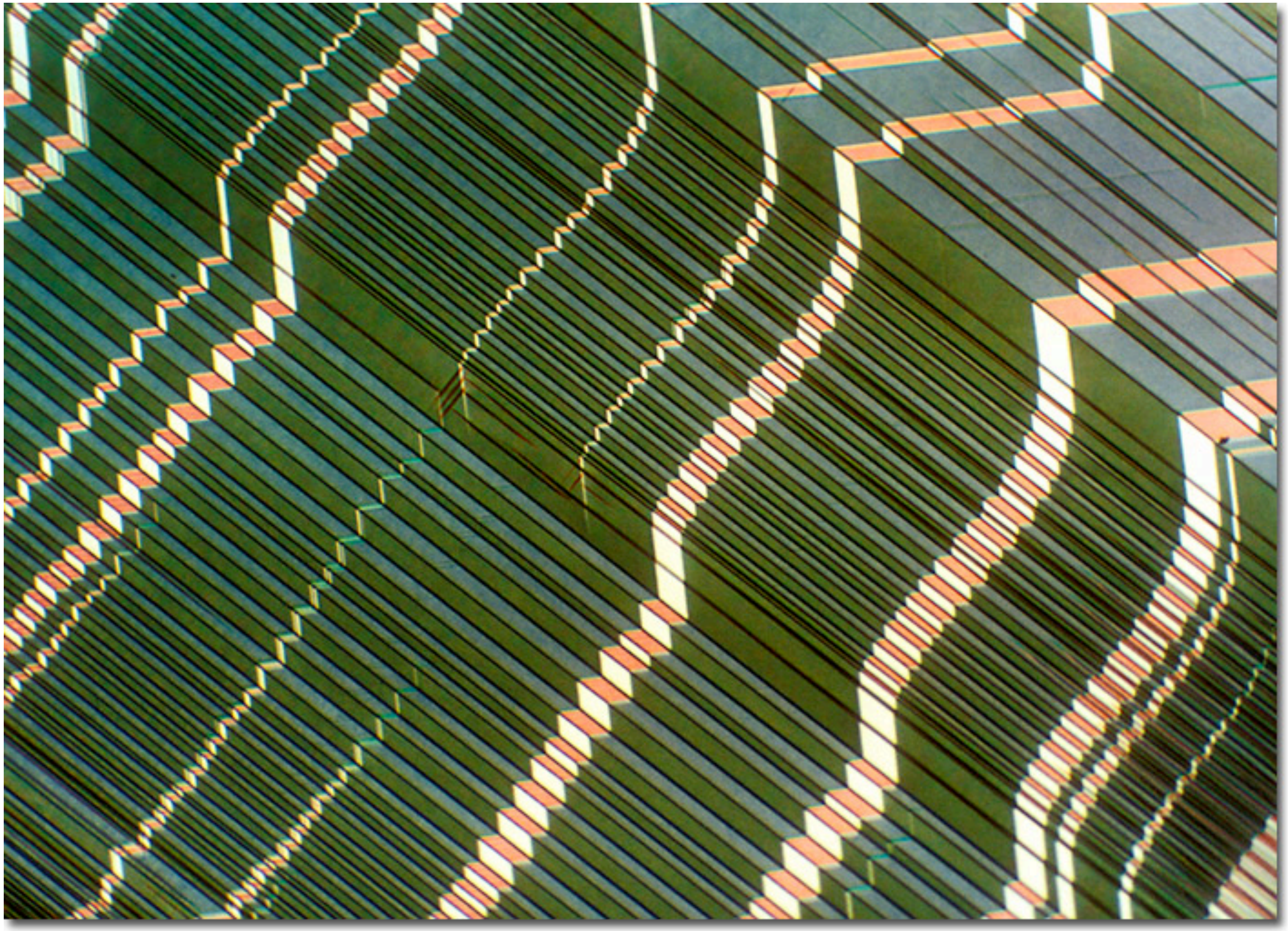
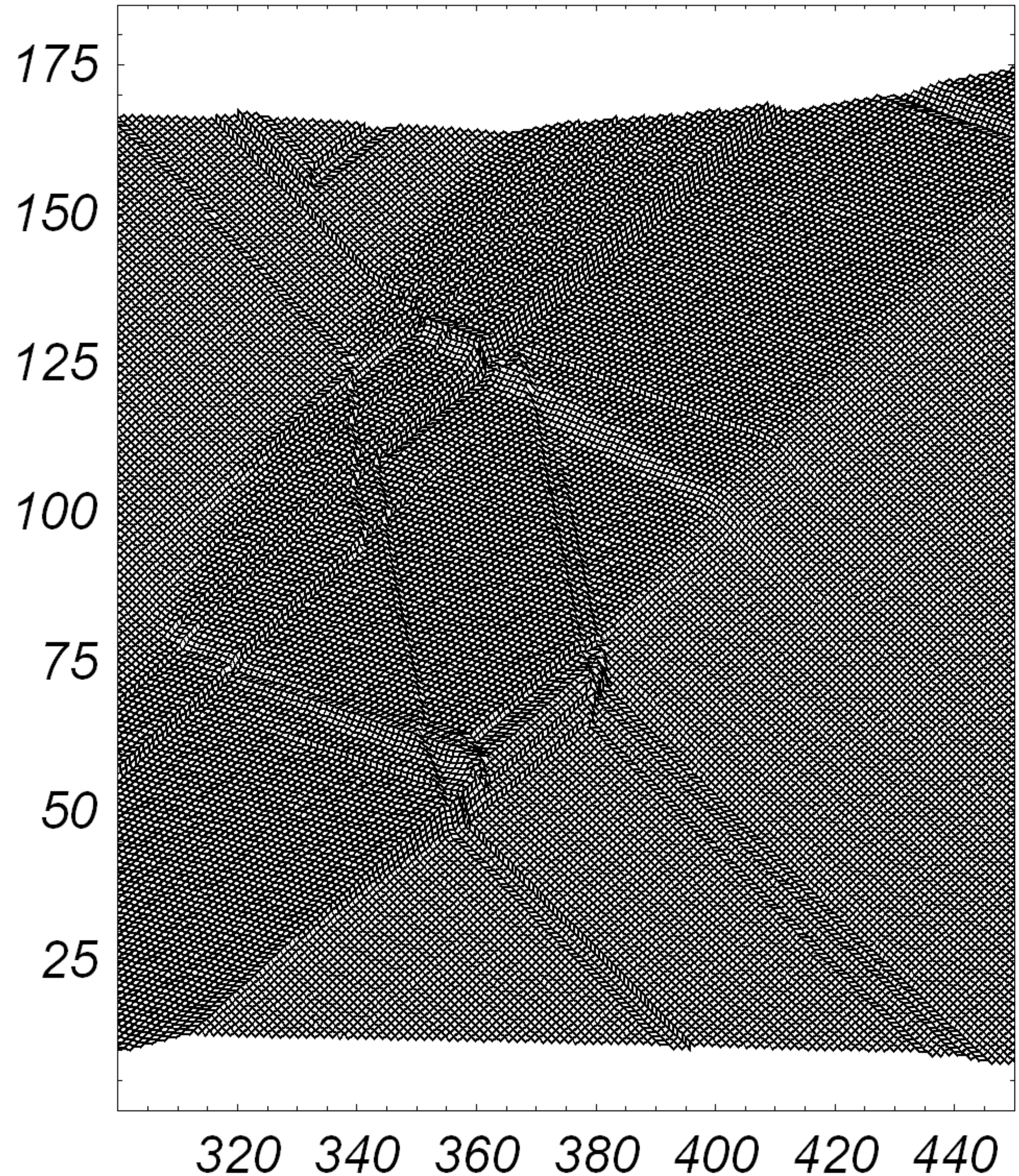
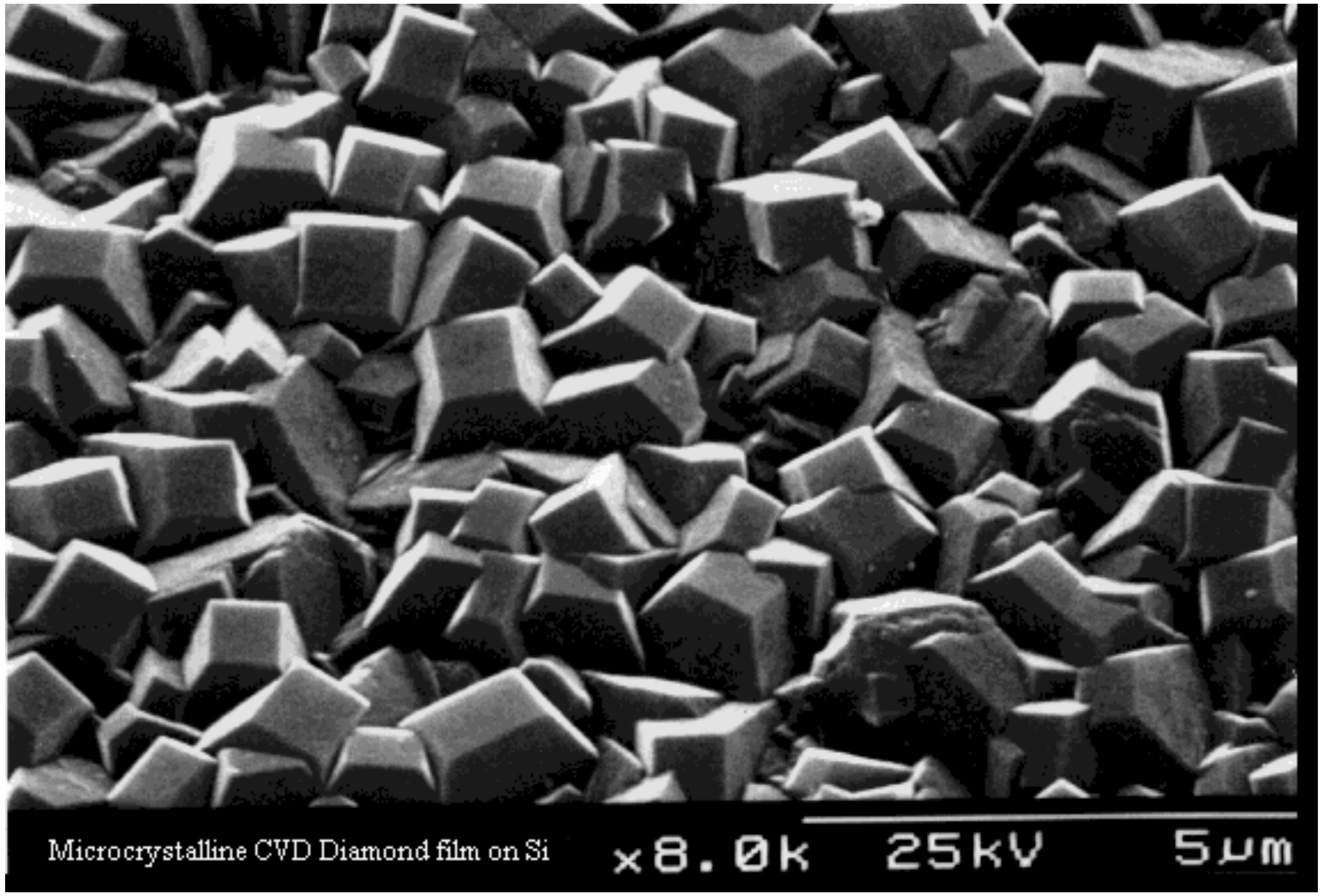
mechanical twinning



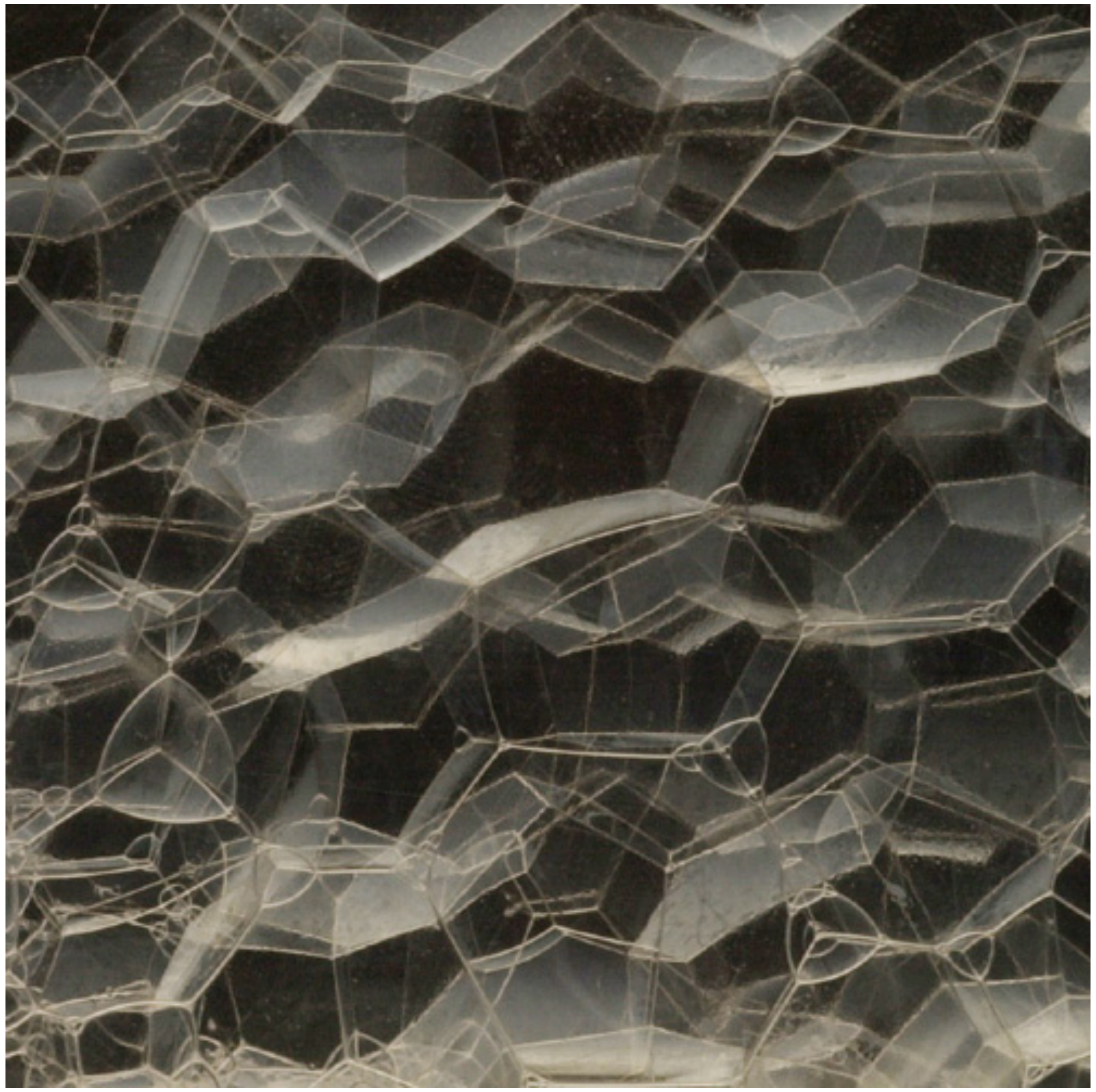
from http://www.magnet.fsu.edu/education/tutorials/gallery/lao.html displacive transformation



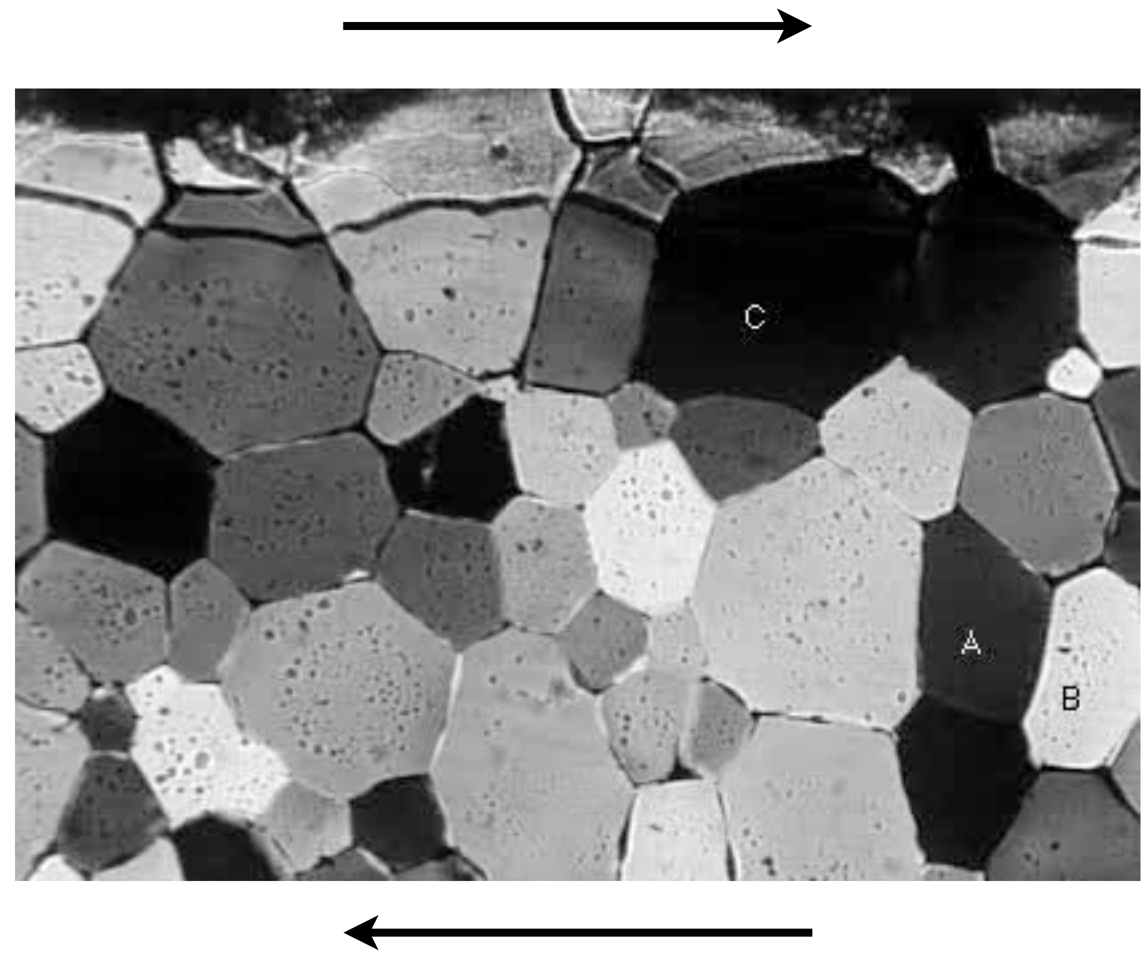
polycrystalline surface



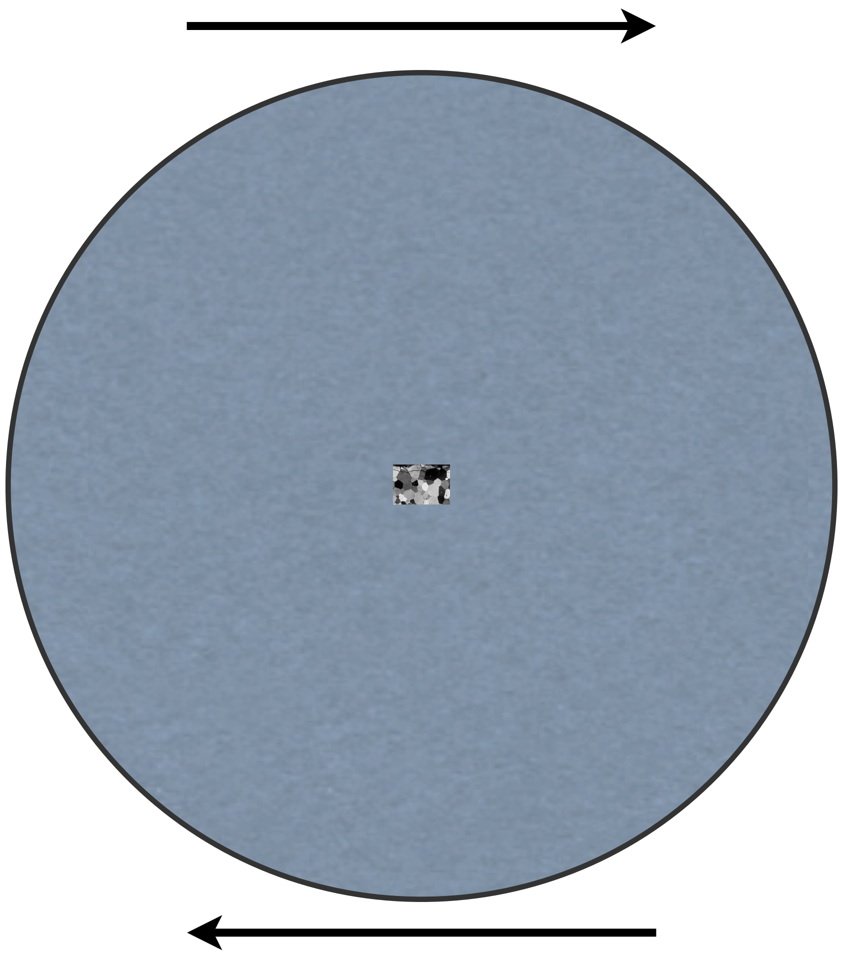
3D soap foam



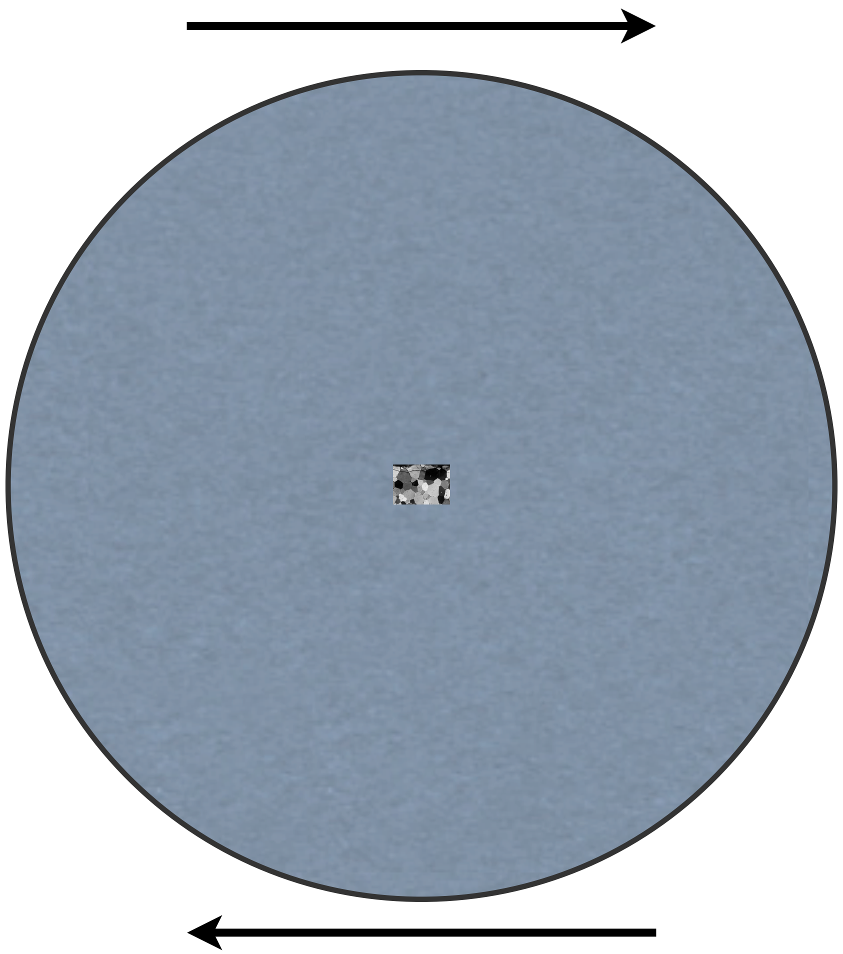
from www.msm.cam.ac.uk


C
A
B

plastic deformation of crystals

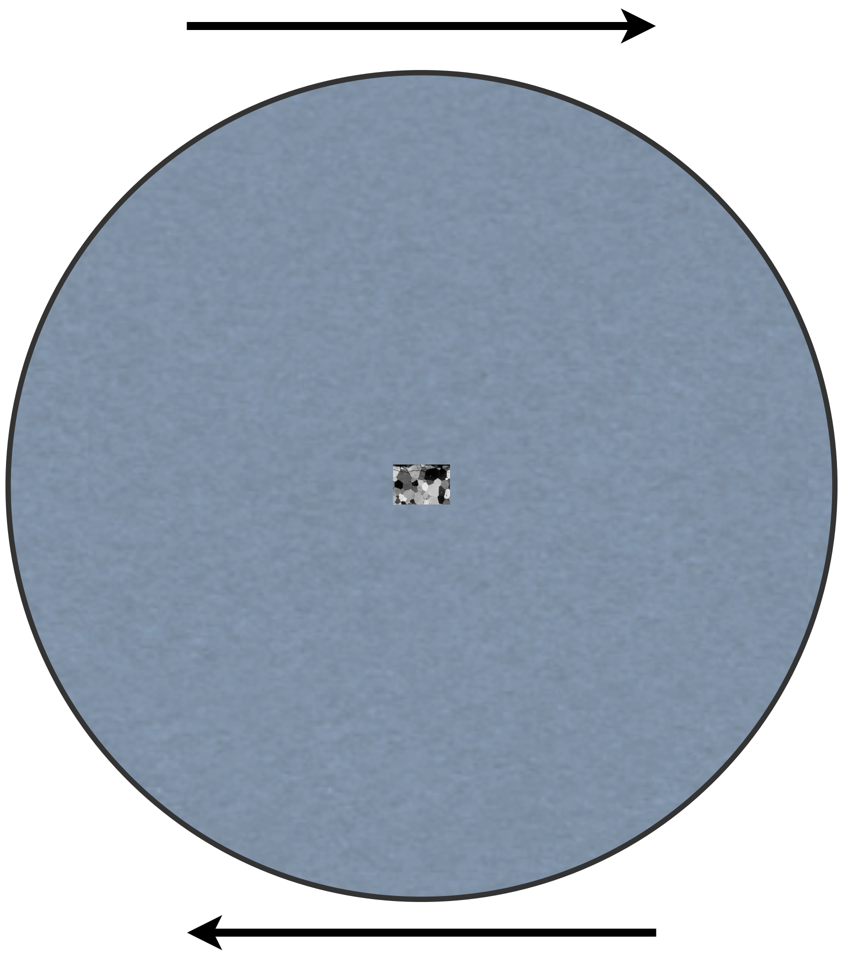


macroscopic response is isotropic only for random distribution of crystallite orientations



macroscopic response is isotropic only for random distribution of crystallite orientations

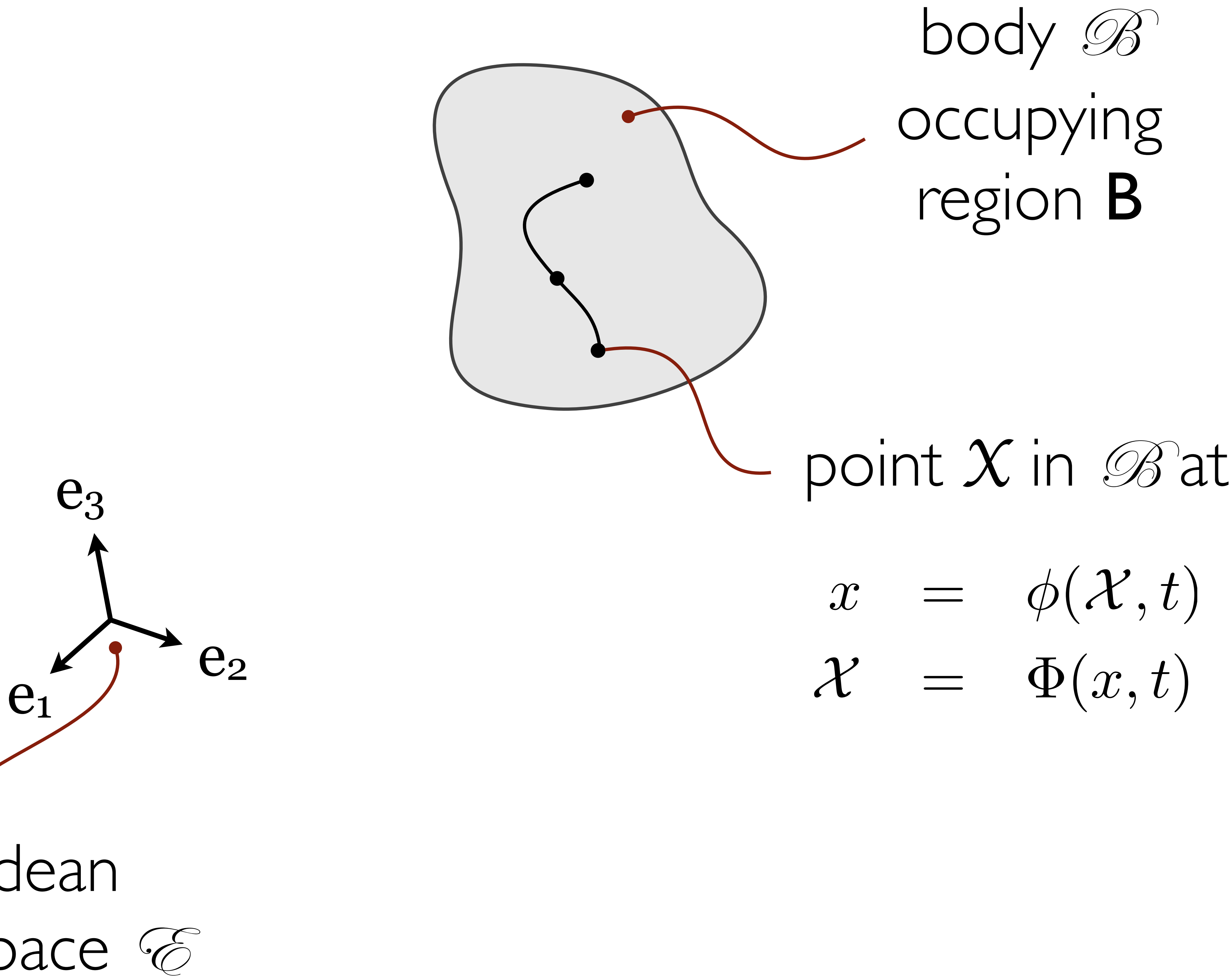
texture causes anisotropy
plastic deformation alters texture



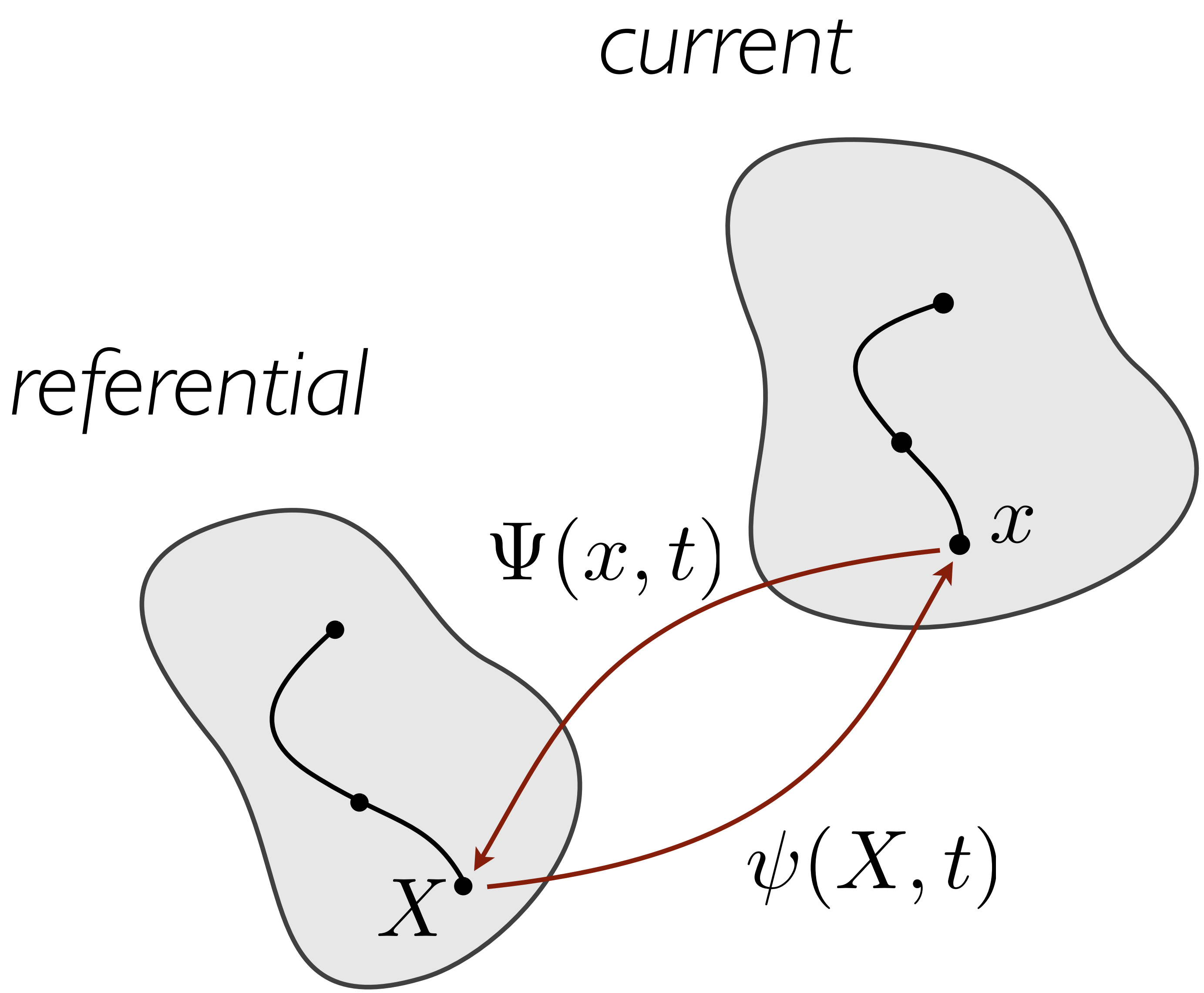
PART II

Background in
Continuum Mechanics

basic kinematics



$$
\begin{aligned}
x &= \phi(\mathcal{X}, t) \\
\mathcal{X} &= \Phi(x, t)
\end{aligned}
$$



define a *referential* configuration:

$$X = \pi(\mathcal{X})$$
$$\mathcal{X} = \Pi(X)$$

such that:

$$x = \phi(\Pi(X),t) = \psi(X,t)$$
$$X = \pi(\Phi(x,t)) = \Psi(x,t)$$

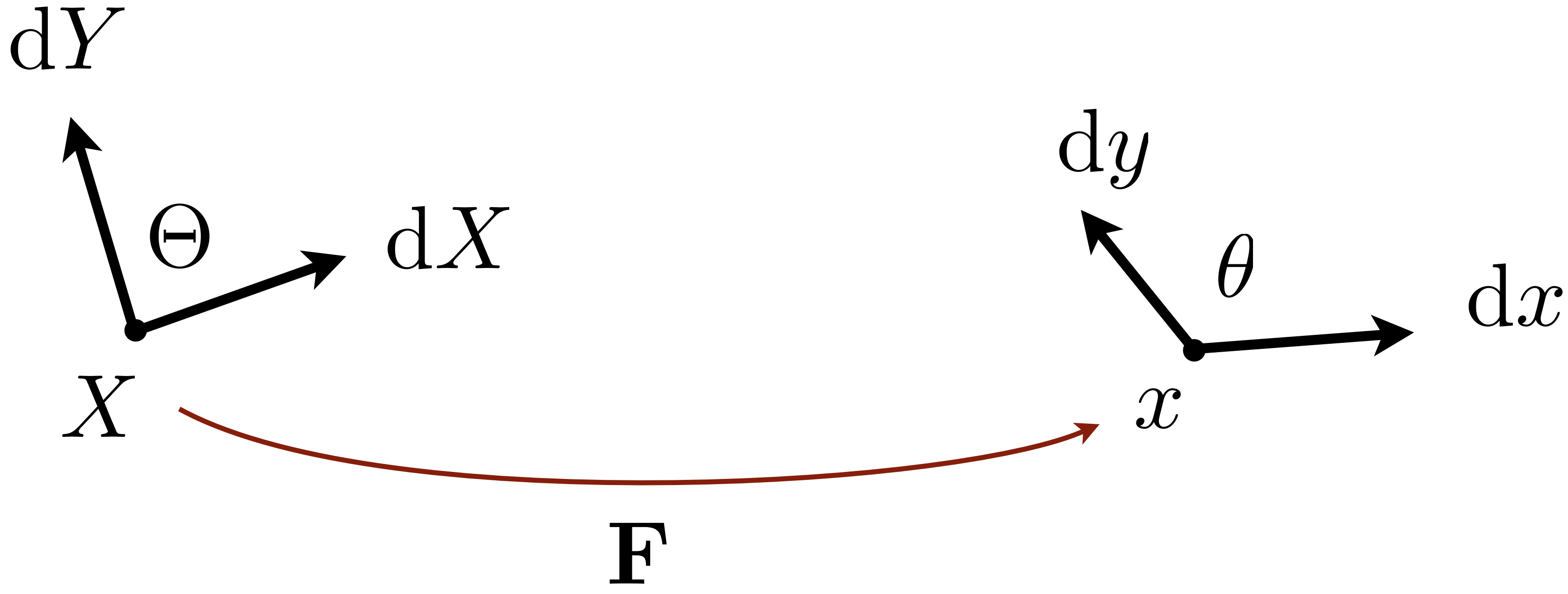
local expansion of deformation map:

$$
\begin{aligned}
x &= \psi(X) + \operatorname{Grad}\psi(X)|_{X_0}\,(X - X_0) + o(X - X_0) \\
&= x_0 + \operatorname{Grad}x|_{x_0}\,(X - X_0) + o(X - X_0) \\
\mathrm{d}x &= \operatorname{Grad}x\,\mathrm{d}X + o(\mathrm{d}X) \\
\mathrm{d}x &= \mathbf{F}\,\mathrm{d}X
\end{aligned}
$$

deformation gradient:

$$
\begin{aligned}
\mathbf{F}(X,t) &= (\operatorname{Grad}\psi)(X,t) \\
\mathbf{F}^{-1}(x,t) &= (\operatorname{grad}\Psi)(x,t)
\end{aligned}
$$

strain measures



strain measures: change of length

$$\mathrm{d}x = \mathbf{l}\,\mathrm{d}s$$
$$\mathrm{d}X = \mathbf{L}\,\mathrm{d}S$$

$$\begin{aligned} l\,\mathrm{d}s \cdot l\,\mathrm{d}s &= \mathbf{F}L\,\mathrm{d}S \cdot \mathbf{F}L\,\mathrm{d}S \\ \left(\frac{\mathrm{d}s}{\mathrm{d}S}\right)^2 &= \mathbf{F}L \cdot \mathbf{F}L \\ &= L \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}L \\ &= L \cdot \mathbf{C}L \end{aligned}$$

right *Cauchy–Green* tensor

$$\begin{aligned} L\,\mathrm{d}S \cdot L\,\mathrm{d}S &= \mathbf{F}^{-1}l\,\mathrm{d}s \cdot \mathbf{F}^{-1}l\,\mathrm{d}s \\ \left(\frac{\mathrm{d}s}{\mathrm{d}S}\right)^2 &= \left(\mathbf{F}^{-1}l \cdot \mathbf{F}^{-1}l\right)^{-1} \\ &= \left(l \cdot \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}L\right)^{-1} \\ &= \left(l \cdot \left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^{-1} L\right)^{-1} \\ &= \left(l \cdot \mathbf{B}^{-1}l\right)^{-1} \end{aligned}$$

left *Cauchy–Green* tensor

strain measures: change of angle

$$\begin{aligned} \mathrm{d}y &= \mathbf{m}\,\mathrm{d}u \\ \mathrm{d}Y &= \mathbf{M}\,\mathrm{d}U \end{aligned}$$

$$\begin{aligned} \cos\theta &= l \cdot m \\ &= \mathbf{F}L\,\frac{\mathrm{d}S}{\mathrm{d}s} \cdot \mathbf{F}M\,\frac{\mathrm{d}U}{\mathrm{d}u} \\ &= L \cdot \mathbf{C}M\,\frac{\mathrm{d}S}{\mathrm{d}s}\,\frac{\mathrm{d}U}{\mathrm{d}u} \end{aligned}$$

polar decomposition of deformation gradient

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

where
$\mathbf{R}$: rotation (proper orthogonal tensor) and
$\mathbf{U}$, $\mathbf{V}$: symmetric, positive definite (stretch) tensor

spectral decomposition of $\mathbf{U}$, $\mathbf{V}$

eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

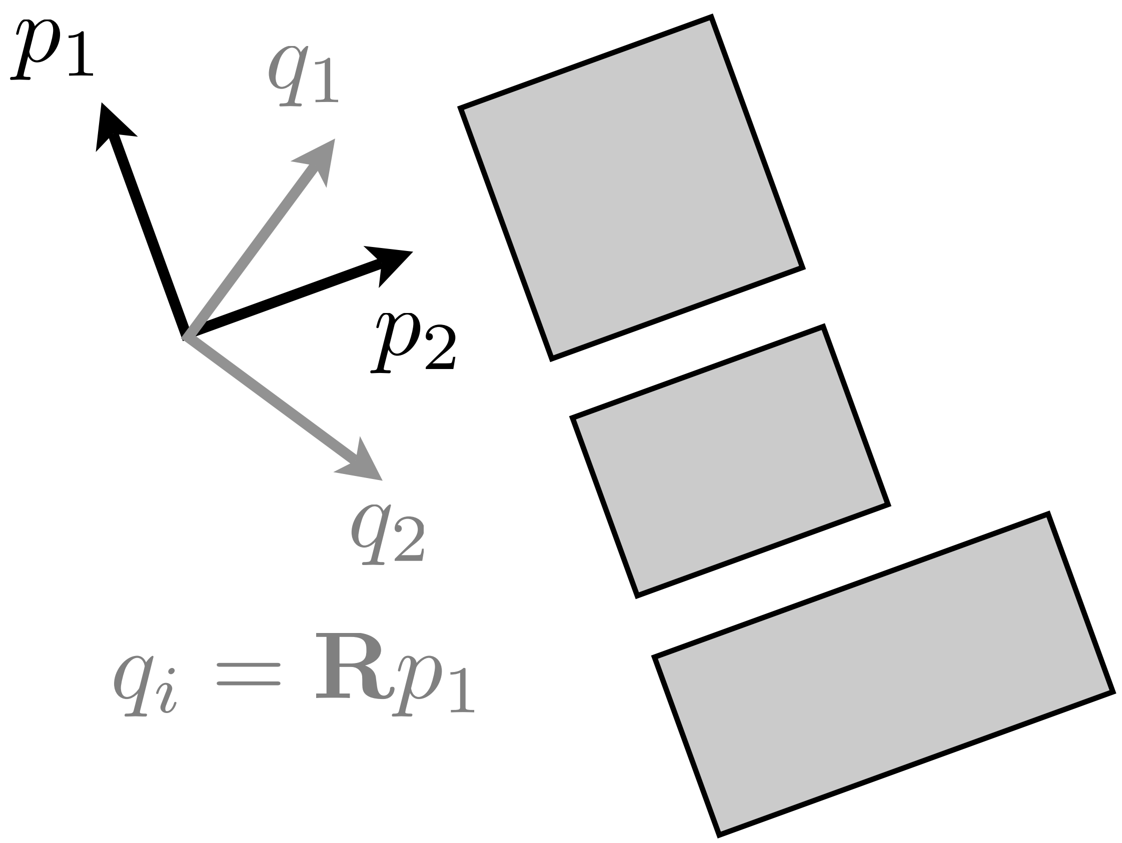
eigenvectors

p_1, p_2, p_3 for $\mathbf{U}$

q_1, q_2, q_3 for $\mathbf{V}$

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, p_i \otimes p_i$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, q_i \otimes q_i$$



relation between polar decomposition and Cauchy–Green tensors $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^{\mathrm{T}}\mathbf{F} \\ &= (\mathbf{R}\mathbf{U})^{\mathrm{T}}(\mathbf{R}\mathbf{U}) \\ &= \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\,\mathbf{U} \\ &= \mathbf{U}^2 \end{aligned} \qquad \begin{aligned} \mathbf{B} &= \mathbf{F}\mathbf{F}^{\mathrm{T}} \\ &= (\mathbf{V}\mathbf{R})(\mathbf{V}\mathbf{R})^{\mathrm{T}} \\ &= \mathbf{V}\,\mathbf{R}\,\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \\ &= \mathbf{V}^2 \end{aligned}$$

interpretation of Cauchy–Green tensors

$\mathbf{C}_{\alpha\beta}$	is the product of stretches along base vectors	E_α, E_β
	and the scalar product	$e_\alpha \cdot e_\beta$

relation of right Cauchy–Green tensor to small strain

$$
\begin{aligned}
\mathbf{C} &= \mathbf{F}^{\mathrm{T}}\mathbf{F} \\
&= (\mathbf{I} + \mathrm{Grad}\, u)^{\mathrm{T}} (\mathbf{I} + \mathrm{Grad}\, u) \\
&= \mathbf{I} + \mathrm{Grad}\, u + (\mathrm{Grad}\, u)^{\mathrm{T}} + \mathrm{Grad}\, u\, (\mathrm{Grad}\, u)^{\mathrm{T}} \\
&\approx \mathbf{I} + (\mathrm{Grad}\, u)_{\mathrm{sym}} \\
&= \mathbf{I} + 2\epsilon
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2}(\mathbf{C} - \mathbf{I}) \\
&= \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \\
&= \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})
\end{aligned}
$$

velocity gradient

$$
\begin{aligned}
\mathbf{L} &= \operatorname{grad} v \\
&= \operatorname{grad} \dot{x} \\
&= (\operatorname{Grad} \dot{x})\, \mathbf{F}^{-1} \\
&= \left(\operatorname{Grad} \left[\frac{\partial}{\partial t}\, x \right] \right) \mathbf{F}^{-1} \\
&= \left(\frac{\partial}{\partial t}\, [\operatorname{Grad} x] \right) \mathbf{F}^{-1} \\
&= \left(\frac{\partial}{\partial t}\, \mathbf{F} \right) \mathbf{F}^{-1} \\
&= \dot{\mathbf{F}}\, \mathbf{F}^{-1}
\end{aligned}
$$

material time derivatives

$$\begin{aligned} \frac{\partial}{\partial t}(\mathrm{d}x) &= \frac{\partial}{\partial t}(\mathbf{F}\,\mathrm{d}X) \\ &= \frac{\partial \mathbf{F}}{\partial t}\,\mathrm{d}X + \mathbf{F}\,\frac{\partial \mathrm{d}X}{\partial t} \\ &= \dot{\mathbf{F}}\,\mathrm{d}X \\ &= \mathbf{L}\mathbf{F}\,\mathrm{d}X \\ (\mathrm{d}x)^{\boldsymbol{\cdot}} &= \mathbf{L}\,\mathrm{d}x \end{aligned}$$

material time derivatives

$$
\begin{aligned}
\frac{\partial}{\partial t}\,(\mathrm{d}x) &= \mathbf{L}\,\mathrm{d}x \\
\frac{\partial}{\partial t}\,(l\,\mathrm{d}s) &= \mathbf{L}\,l\,\mathrm{d}s \\
\frac{\partial l}{\partial t}\,\mathrm{d}s + l\,\frac{\partial \mathrm{d}s}{\partial t} &= \mathbf{L}\,l\,\mathrm{d}s \\
l\cdot\dot{l}\,\mathrm{d}s + l\cdot l\,(\mathrm{d}s)^{\cdot} &= l\cdot\mathbf{L}\,l\,\mathrm{d}s \\
(\mathrm{d}s)^{\cdot} &= l\cdot\mathbf{L}\,l\,\mathrm{d}s
\end{aligned}
$$

material time derivatives

$$\begin{aligned} \dot{l}\,\mathrm{d}s + l\,(\mathrm{d}s)^{\cdot} &= \mathbf{L}\,l\,\mathrm{d}s \\ \dot{l}\,\mathrm{d}s + l\,(l \cdot \mathbf{L}\,l\,\mathrm{d}s) &= \mathbf{L}\,l\,\mathrm{d}s \\ \dot{l} &= \mathbf{L}\,l - (l \cdot \mathbf{L}\,l)\,l \end{aligned}$$

material time derivatives

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\cos\theta\right) &= \frac{\partial}{\partial t}\left(l\cdot m\right) \\
-\sin\theta\,\dot{\theta} &= \dot{l}\cdot m + l\cdot\dot{m} \\
\dot{\theta} &= \left|l\times m\right|^{-1} \\
&\quad \left[\left\{l\cdot\mathbf{L}l + m\cdot\mathbf{L}m\right\}\left(l\cdot m\right) - l\cdot\left\{\left(\mathbf{L}+\mathbf{L}^{\mathrm{T}}\right)m\right\}\right]
\end{aligned}
$$

stretching and spin

$\mathbf{F}$ specifies changes of size and shape while
$\mathbf{L}$ describes the *rate* of those changes

additive decomposition of $\mathbf{L}$

$$\begin{aligned} \mathbf{L} &= \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathrm{T}}\right) \\ &= \quad\mathbf{D} \quad\quad + \quad\quad \mathbf{W} \end{aligned}$$

symmetric   skew-symmetric

stretching and spin

interpretation of $\mathbf{D}$ and $\mathbf{W}$ using

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

$$\begin{aligned}
\rightarrow \quad \mathbf{L} &= (\mathbf{R}\mathbf{U})\dot{}(\mathbf{R}\mathbf{U})^{-1} \\
&= (\dot{\mathbf{R}}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}})(\mathbf{U}^{-1}\mathbf{R}^{-1}) \\
&= \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^T
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}\mathbf{R}^T &= \mathbf{I} \\
\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T &= \mathbf{0} \\
\dot{\mathbf{R}}\mathbf{R}^T + \left(\dot{\mathbf{R}}\mathbf{R}^T\right)^T &= \mathbf{0}
\end{aligned}$$

stretching and spin

interpretation of $\mathbf{D}$ and $\mathbf{W}$ using

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

$$\rightarrow \quad \mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^T$$

$$\mathbf{D} = \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^T$$

$$\mathbf{W} = \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^T + \dot{\mathbf{R}}\mathbf{R}^T$$

stretching and spin

suppose reference configuration equals current configuration:

$$\mathbf{F} = \mathbf{R} = \mathbf{U}\ (= \mathbf{V}) = \mathbf{I}$$

rate of change of stretch ...

$$\mathbf{D} = \frac{1}{2}\mathbf{R}\left[\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}\right]\mathbf{R}^T = \dot{\mathbf{U}}_0$$

$$\mathbf{W} = \frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}\right)\mathbf{R}^T + \dot{\mathbf{R}}\mathbf{R}^T = \dot{\mathbf{R}}_0$$

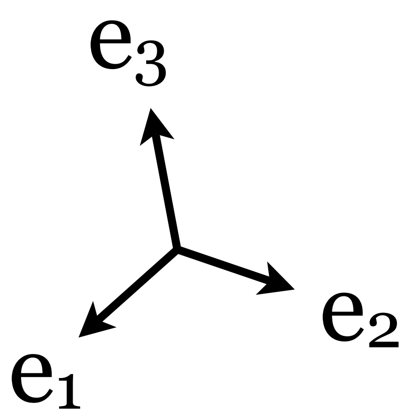
rate of change of rotation while passing through current configuration

geometric interpretation of stretching and spin around point in current configuration

$$\frac{\dot{(ds)}}{ds} = \lambda \cdot \mathbf{L} \cdot \lambda$$

$$= \lambda \cdot \mathbf{D} \cdot \lambda$$

$\mathbf{D}_{ii}$: rate of extension along base vector $\mathbf{e}_i$



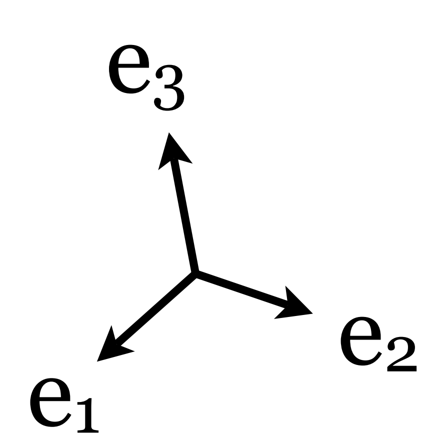
geometric interpretation of stretching and spin around point in current configuration

!ú = |l ! m|! 1

! {l áL l + m áL m} (l ám) " l á "# L + L T $ m %

with l ám = 0

" 1/2 !ú = l áD m



Dij : half the rate of decrease in angle between base vectors **e**i and **e**j

geometric interpretation of stretching and spin around point in current configuration

spectral decomposition of $\mathbf{D}$ results in three principal stretchings ν_i and their respective (orthogonal) axes $\mathbf{r}_\mathrm{i}$

rate of angular change is zero, thus principal axes perform (rigid) rotation

geometric interpretation of stretching and spin around point in current configuration

principal axes $\mathbf{r}_i$ perform (rigid) rotation

$$\dot{\mathbf{l}} = \mathbf{L}\,\mathbf{l} - (\mathbf{l}\cdot\mathbf{L}\,\mathbf{l})\,\mathbf{l}$$

$$\forall \quad \dot{\mathbf{r}}_i = (\mathbf{D} + \mathbf{W})\,\mathbf{r}_i - (\mathbf{r}_i\cdot\mathbf{D}\,\mathbf{r}_i)\,\mathbf{r}_i$$

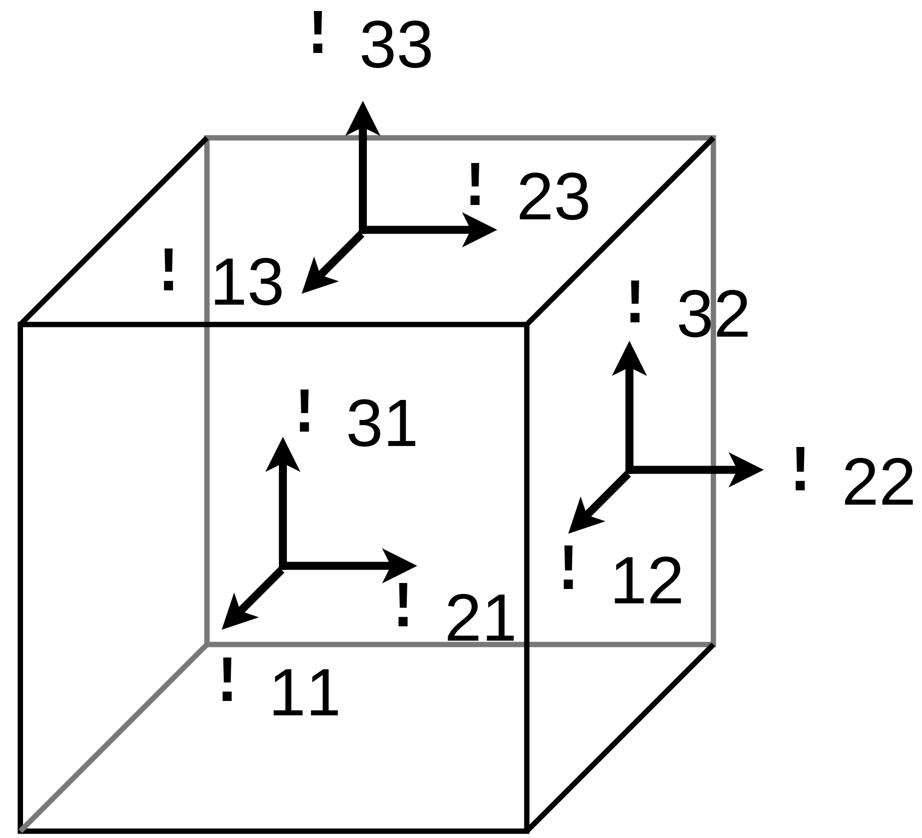
$$= \lambda_i\,\mathbf{r}_i + \mathbf{W}\,\mathbf{r}_i - (\mathbf{r}_i\cdot\lambda_i\,\mathbf{r}_i)\,\mathbf{r}_i$$

$$= \mathbf{W}\,\mathbf{r}_i$$

$$= \mathbf{w} \times \mathbf{r}_i$$

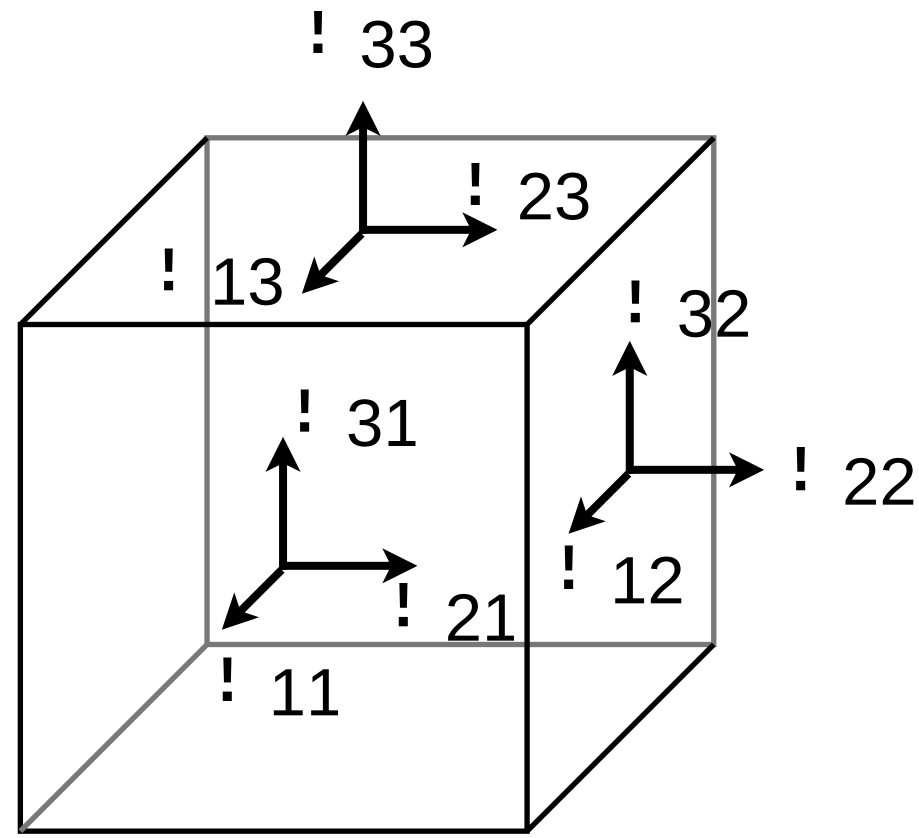
Cauchy stress tensor

$$t_{(n)} = ! \; n$$



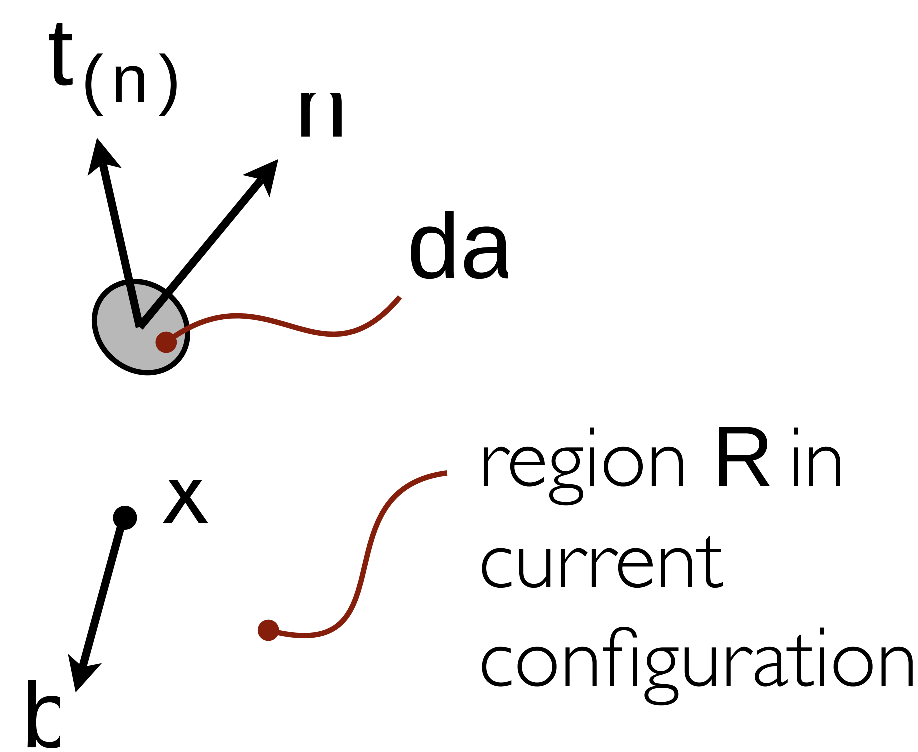
angular momentum balance

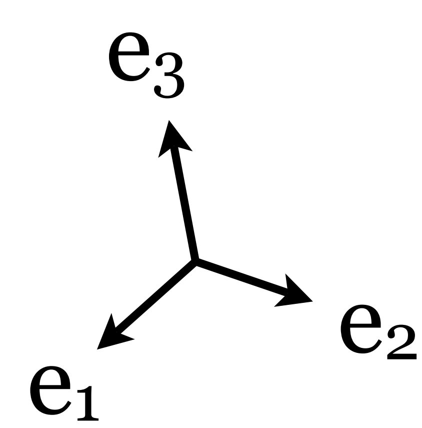
$$!^{T} = !$$



linear momentum balance

$$\frac{d}{dt}\int_R \rho\,\dot{x}\,dv = \int_R \rho\,b\,dv + \int_{\partial R} t_{(n)}\,da$$

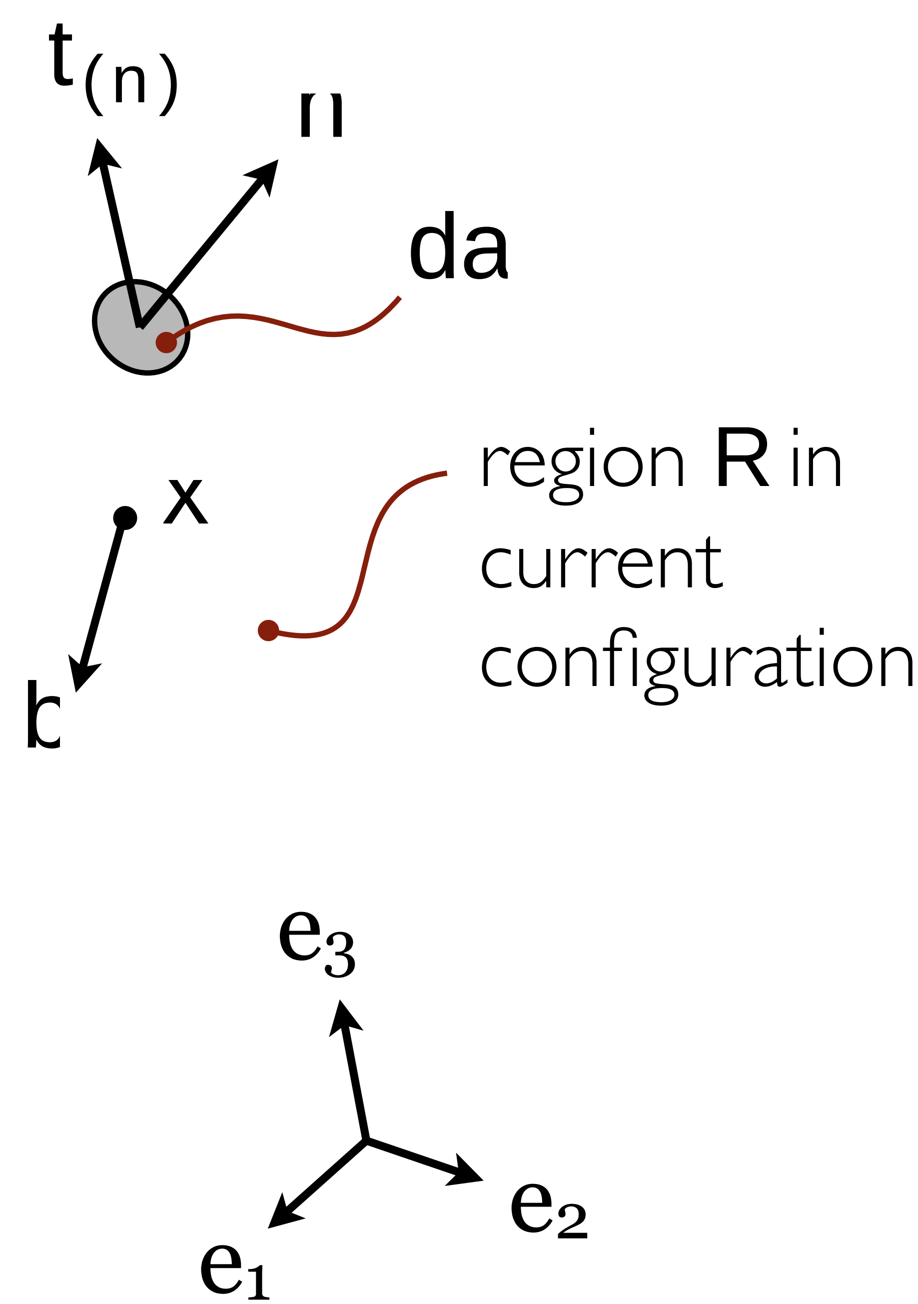




linear momentum balance

$$\frac{d}{dt}\int_R \rho\,\dot{x}\,dv = \int_R \rho\, b\,dv + \int_{\partial R} t_{(n)}\,da$$

$$\int_R \operatorname{div} \sigma\, dv = 0$$



stress measures

Cauchy $!$

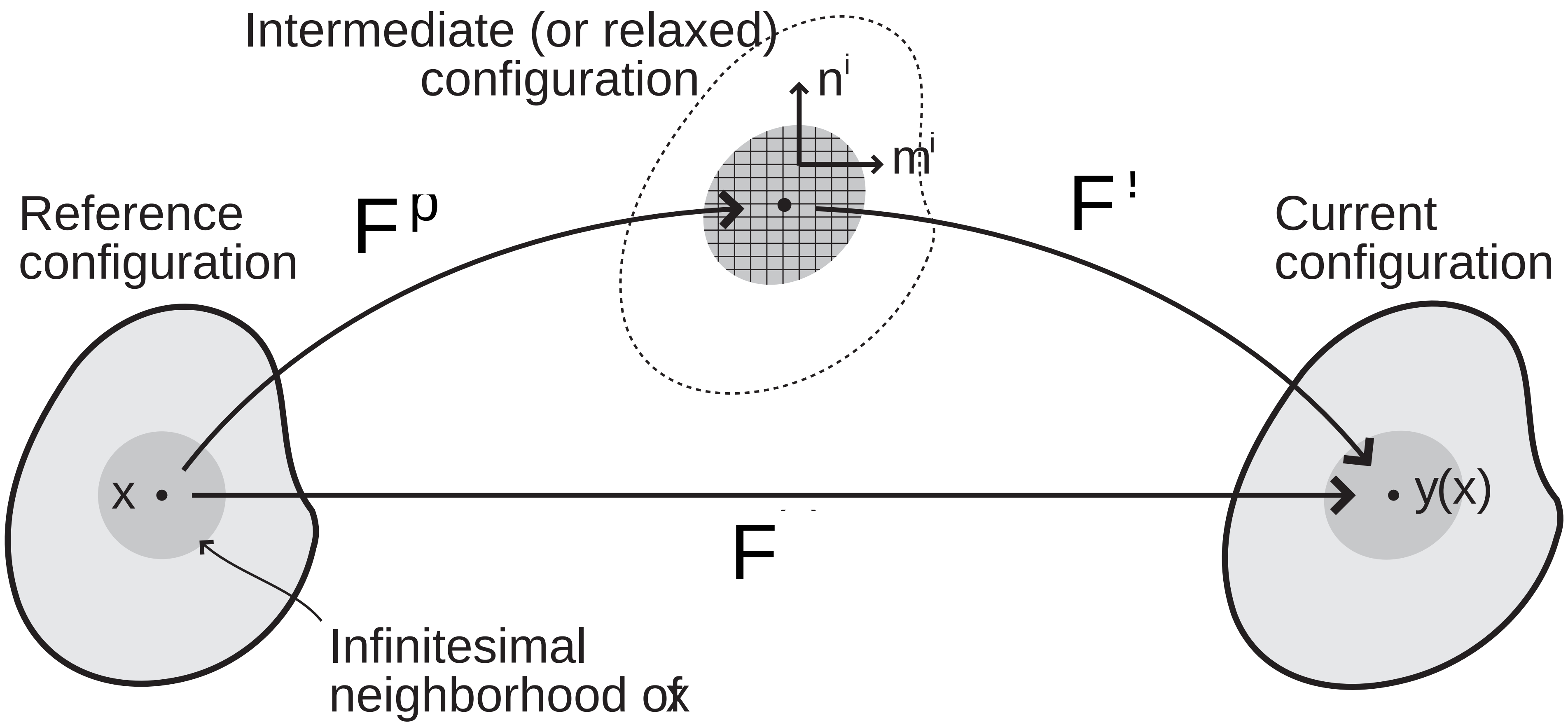
first Piola–Kirchhoff $\mathbf{P} = J\, ! \, \mathbf{F}^{!\, T} = \mathbf{F}\,\mathbf{S}$

second Piola–Kirchhoff $\mathbf{S} = J\, \mathbf{F}^{!\, 1}\, ! \, \mathbf{F}^{!\, T}$

$$
\begin{aligned}
P_{int} &= \int_v \sigma : L \, dv = \int_v \sigma : D \, dv \\
&= \int_V P : \dot{F} \, dV \\
&= \int_V S : \dot{E} \, dV
\end{aligned}
$$

$A : (BC) = (B^T A) : C = (AC^T) : B$ useful for derivation

finite strain plasticity



finite strain plasticity

multiplicative decomposition of deformation gradient

$$F = F^{!} F^{p} \quad !" \quad F^{!} = F F^{p\,"\,1}$$

finite strain plasticity

multiplicative decomposition of deformation gradient

$$F = F^{e} F^{p} \quad \Rightarrow \quad F^{e} = F \, F^{p\,-1}$$

plastic velocity gradient

$$L^{p} = \dot{F}^{p} F^{p\,-1} \quad \Rightarrow \quad \dot{F}^{p} = L^{p} F^{p}$$

multiplicative decomposition of deformation gradient

$$\mathbf{F} = \mathbf{F}^e\,\mathbf{F}^p \quad \Rightarrow \quad \mathbf{F}^e = \mathbf{F}\,\mathbf{F}^{p\,-1}$$

plastic velocity gradient

$$\mathbf{L}^p = \dot{\mathbf{F}}^p\,\mathbf{F}^{p\,-1} \quad \Rightarrow \quad \dot{\mathbf{F}}^p = \mathbf{L}^p\,\mathbf{F}^p$$

elastic Green's Lagrangian strain

$$\mathbf{E} = \frac{1}{2}\left[\mathbf{F}^{e\,T}\mathbf{F}^e - \mathbf{I}\right]$$

multiplicative decomposition of deformation gradient

$$F = F^e F^p \quad \Rightarrow \quad F^e = F F^{p-1}$$

plastic velocity gradient

$$L^p = \dot{F}^p F^{p-1} \quad \Rightarrow \quad \dot{F}^p = L^p F^p$$

elastic Green's Lagrangian strain

$$E = \frac{1}{2}\left[F^{eT} F^e - I \right]$$

work-conjugate second Piola–Kirchhoff stress

$$S = C : E = \frac{1}{2} C : \left[F^{p-T} F^T F F^{p-1} - I \right]$$

finite strain plasticity

$! = t + ! t$

fully-implicit formulation of rate of change of plastic deformation gradient

$$\dot{F}^p = \frac{F^p(!) ! F^p(t)}{! t} = L^p(!)F^p(!)$$

$$! = t + ! \, t$$

fully-implicit formulation of rate of change of plastic deformation gradient

$$\dot{F}^p = \frac{F^p(!) \, ! \, F^p(t)}{! \, t} = L^p(!) F^p(!)$$

after rearranging

$$F^{p! \, 1}(!) = F^{p! \, 1}(t) \, [I \, ! \, ! \, t \, L^p(!)]$$

$$F^{p! \, T}(!) = I \, ! \, ! \, t \, L^{pT}(!) \; F^{p! \, T}(t)$$

combination of

$$S = C : E = \frac{1}{2} C : \left[F^{p-T} F^{T} F F^{p-1} ! \; I \right]$$

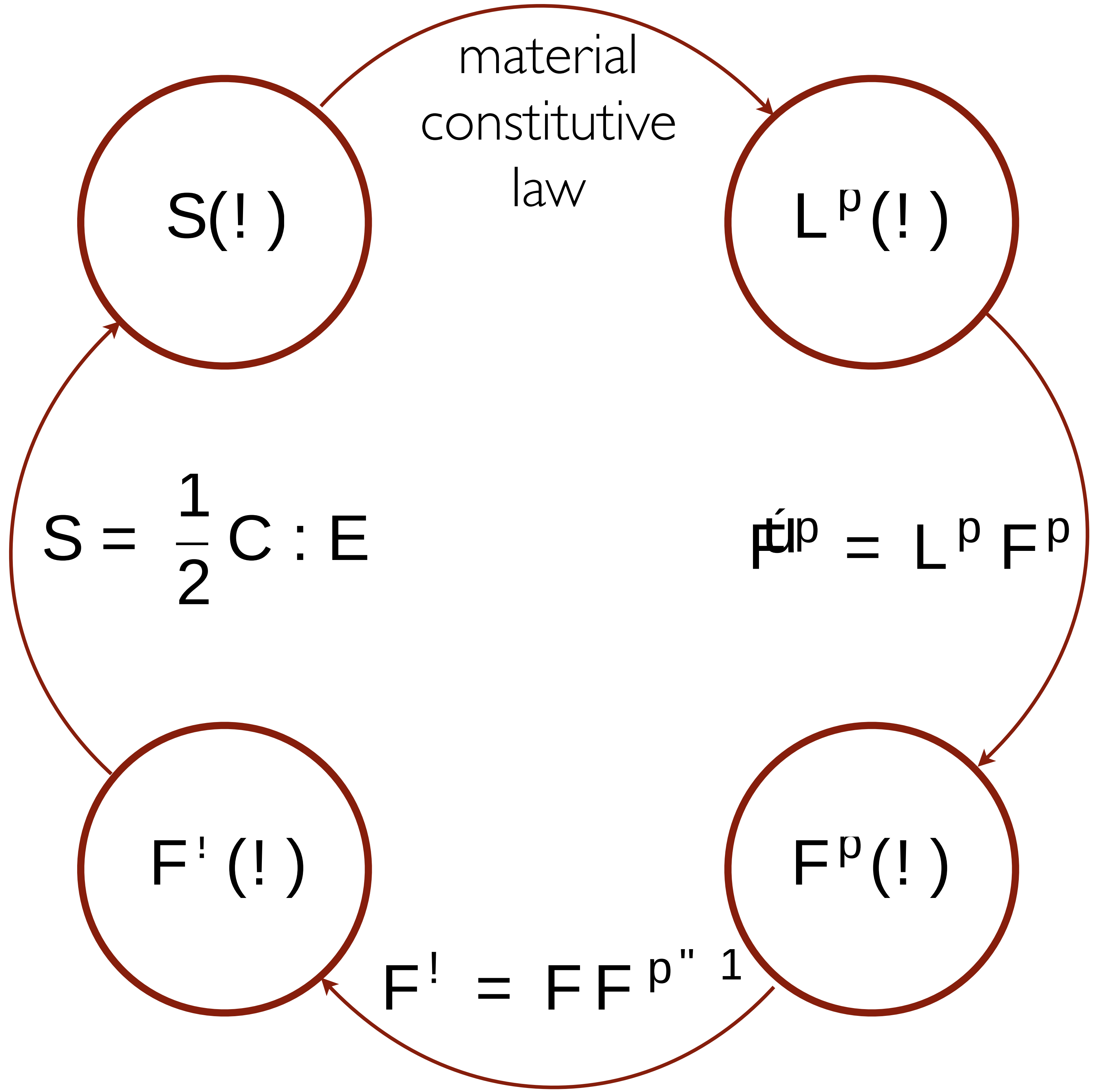
with

$$F^{p-1}(!) = F^{p-1}(t)\,[I \; ! \; ! \; t \, L^{p}(!)]$$

$$F^{p-T}(!) = \left[I \; ! \; ! \; t \, L^{pT}(!) \right] F^{p-T}(t)$$

$$S(!) = \frac{1}{2} C : \left[\underbrace{\left[I \; ! \; ! \; t \, L^{pT}(!) \right]}_{B^{T}} \underbrace{F^{p-T}(t) F^{T}(!) F(!) F^{p-1}(t)}_{A} \underbrace{[I \; ! \; ! \; t \, L^{p}(!)]}_{B} ! \; I \right]$$

elasto-plastic consistency



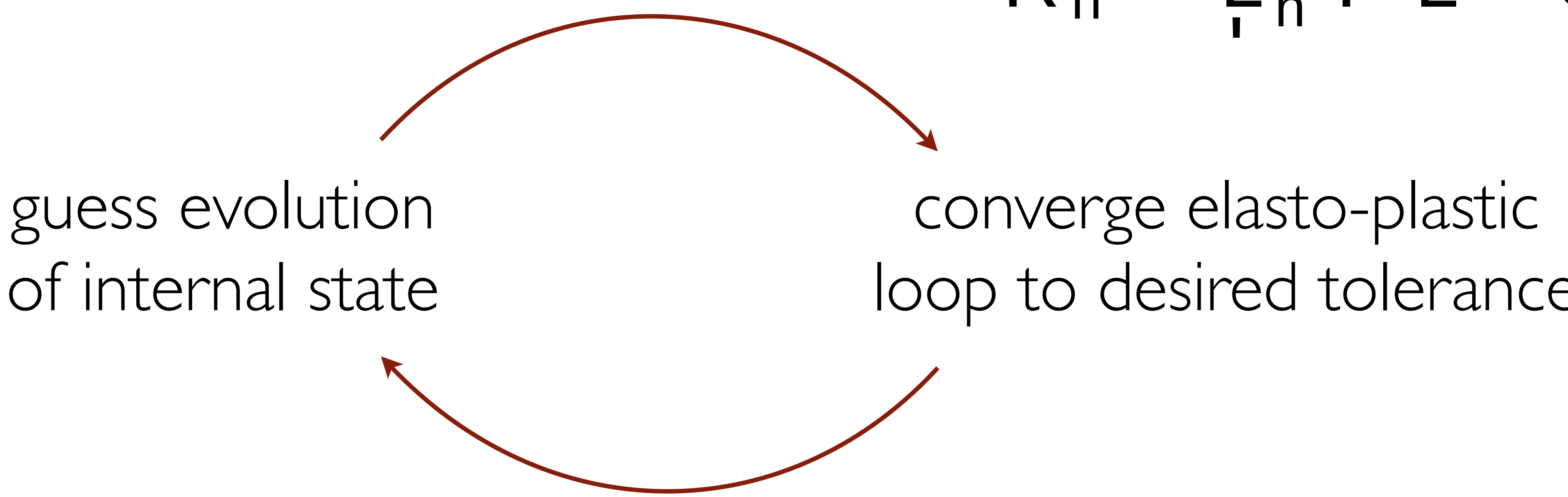
material
constitutive
law

depends on stress $\mathbf{S}$ and
internal state variables $\mathbf{s}$

$$L^{p}(!) = L^{p}(S, s)$$
$$\dot{s}(!) = \dot{s}(S, s)$$

two-level predictor–corrector scheme

$$R_n = L^p_n \,!\; L^p \;\; S \;\; L^p_n \;\cdots$$



$$r_n = \dot{s}_n \;!\;\; s(t)\;\ddot{}\;\;!\;\;!\;\; t\, s(S_n, s_n)$$

residuum of elasto-plastic loop

$$R_n = L^p_n \,!\, \; L^p \;\; S \;\; L^p_n \,\cdots$$

Newton–Raphson correction

$$L^p_{n+1} = L^p_n \,!\, \; \# \left.\frac{! R_n}{! L^p}\right\$_{L^p_n} \& ^{\%! \; 1} : R_n$$

$$\frac{! R_{ij}}{! L^{p}_{kl}} = \frac{! L^{p}_{ij}}{! L^{p}_{kl}} ! \frac{! L^{p}_{ij}}{! S_{mn}} \frac{! S_{mn}}{! L^{p}_{kl}}$$

$$= "_{ik} "_{jl} ! \frac{! L^{p}_{ij}}{! S_{mn}} \frac{! S_{mn}}{! L^{p}_{kl}}$$

$$\frac{! S}{! L^{p}} =$$

$$S_{,L^{p}} = \frac{1}{2} " C : \# B^{T} A B ! I \$\%_{,L^{p}}$$

$$\& = \frac{1}{2} C : \# B^{T} A B ! I \$_{,L^{p}} + \# B^{T} A B ! I \$: C_{,L^{p}} '$$

$$= \frac{1}{2} C : \left(B^{T}_{,L^{p}} A B + B^{T} A B_{,L^{p}} \right)$$

$$= ! \frac{! t}{2} C : \left(L^{pT}_{,L^{p}} A B + B^{T} A L^{p}_{,L^{p}} \right)$$

$$\frac{!\,S}{!\,L^{p}} =$$

$$S_{,L^{p}} = \frac{1}{2}\text{"}\, C : \#\, B^{T} A B \; ! \; I \,\$\%_{,L^{p}}$$

$$= \frac{1}{2}\,\&\, C : \#\, B^{T} A B \; ! \; I \,\$_{,L^{p}} + \#\, B^{T} A B \; ! \; I \,\$: C_{,L^{p}}\,'$$

$$= \frac{1}{2} C : \left(B^{T}_{,L^{p}} A B + B^{T} A B_{,L^{p}} \right)$$

$$= \; ! \; \frac{!\;t}{2} C : \left(L^{pT}_{,L^{p}} A B + B^{T} A L^{p}_{,L^{p}} \right)$$

$$\frac{!\,S}{!\,L^{p}} =$$

$$S_{,L^{p}} = \frac{1}{2}\, " \, C : \#\, B^{T} A B \; ! \; I \,\$\%_{,L^{p}}$$

$$= \frac{1}{2}\, \& \, C : \#\, B^{T} A B \; ! \; I \,\$_{,L^{p}} + \#\, B^{T} A B \; ! \; I \,\$: C_{,L^{p}} \, '$$

$$= \frac{1}{2}\, C : \left(B^{T}{}_{,L^{p}} A B + B^{T} A B_{,L^{p}} \right)$$

$$= \; ! \; \frac{!\;t}{2}\, C : \left(L^{pT}{}_{,L^{p}} A B + B^{T} A L^{p}{}_{,L^{p}} \right)$$

$$\frac{!\,S_{ij}}{!\,L^{p}_{kl}} = \; ! \; \frac{!\;t}{2}\, C_{ijmn} \; \frac{!\,L^{pT}_{mq}}{!\,L^{p}_{kl}} A_{qp} B_{pn} + B^{T}_{mp} A_{pq} \frac{!\,L^{p}_{qn}}{!\,L^{p}_{kl}}$$

$$= \; ! \; \frac{!\;t}{2}\, C_{ijmn} \; \$ \, "_{qk} \, "_{ml} \, (AB)_{qn} + (B^{T} A)_{mq} \, "_{qk} \, "_{nl} \, \%$$

$$= \; ! \; \frac{!\;t}{2}\, \$ \, C_{ijln} \, (AB)_{kn} + C_{ijml} \, (B^{T} A)_{mk} \, \%$$